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EVALUATION OF EPHEMERIS REPRESENTATIONS FOR THE MULTIMISSION MODULAR SPACECRAFT (MMS)

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By

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ABSTRACT

Results are presented of an evaluation of several ephemeris representations for the Multimission Modular Spacecraft (MMS). Primary evaluation criteria are the accuracy and onboard computational cost, with particular reference to the Earth Observation Satellite (EOS) mission. Representations in Cartesian coordinates, equinoctial elements, and orbital quaternions are examined. In particular, Fourier-power series representations are evaluated primarily for extrapolation and data compression purposes, and Lagrange and Hermite polynomial representations are evaluated primarily for interpolation purposes.

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SECTION 1 - INTRODUCTION

With the increasing number of spacecraft missions and the accompanying requirement for detailed mission design, the need for standardizing spacecraft design is becoming increasingly critical. As a result, the concept of a Multimission Modular Spacecraft (MMS)[1] has been developed. This spacecraft will use standardized, modularized hardware components and onboard software which will be adaptable to a variety of missions (References 1, 2).

One of the goals of the MMS design is 72-hour autonomy from ground support. Autonomy implies the availability of both orbit and attitude information onboard the spacecraft, primarily to enable the spacecraft to perform attitude control and secondarily for onboard computation of pointing maneuver controls. The attitude control will be performed onboard as follows. An onboard digital processor will compute commands for the attitude control components (such as reaction wheels) based on the current and the target inertial attitudes. (At present, all MMS missions are planned to employ three-axis stabilized spacecraft.) The current inertial attitude will be computed using a digital filter that processes gyro data along with star sensor or Sun sensor data. The target inertial attitude will be transmitted periodically to the onboard processor by the ground station.[2] For star- or Sun-pointing missions, the target attitude is constant except during slewing maneuvers. However, for Earth-pointing missions, the target attitude is orbital position dependent. Therefore, the ground update in this case must include accurate orbital information. Orbital information is also needed onboard the spacecraft for other reasons, which are summarized below.

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[1]Formerly Low Cost Modular Spacecraft (LCMS).

[2]Alternatively, for Earth-pointing missions, it could be computed from an ephemeris determined onboard from the Global Positioning System (GPS) signals (Appendix C and Reference 3).

recurring computations in the attitude control function on EOS, which is performed in terms of quaternions (Reference 4).

Section 3 is devoted to an examination of the relative computational costs of various representations. An estimate is made of the computational times which are required on the NASA Standard Spacecraft Computer (NSSC) for interpolation and for conversion from the selected type of elements to a form suitable for onboard use. Core requirements are also discussed based on the results of Section 2.

Section 4 summarizes the results of this study and presents preliminary conclusions.

missions typified by the Earth Observation Satellite (EOS) with a 700-kilometer altitude and a circular Sun-synchronous orbit, and it must agree with the predicted orbit to within about 10 kilometers for the star- and Sun-pointing missions typified by the Gamma Ray Explorer (GRE) and the Solar Maximum Mission (SMM)[1] with a 500- to 550-kilometer altitude and a 30-degree inclination circular orbit. The desirable span over which the representation must maintain this accuracy is 3 to 4 days, with the capability of a 1-day extrapolation (possibly with reduced accuracy). Appendix A discusses the basis for these accuracy requirements.

In this memorandum, results from an evaluation of several ephemeris representation techniques are presented addressing the following areas:

- Interpolation accuracy
- Extrapolation accuracy
- Onboard computational simplicity and speed
- Onboard core requirement
- Data transmission from the ground (amount and frequency)
- Standardization and adaptability of the onboard software
- Compatibility with orbit determination algorithms using the Global Positioning System (GPS) measurements as input (Reference 3)

Results of an evaluation of the accuracy of several ephemeris representations are presented in Section 2. Schemes that employ classical and equinoctial orbital elements,[2] Cartesian coordinates, and orbital quaternions[3] are examined. The quaternion form of orbit description is motivated by the possibility of reducing the

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[1]Due to a recent scientific data annotation requirement, ephemeris representation accuracies of 100 meters for the cross-track and radial components and 500 meters for the along-track component are needed. This requirement has not been accounted for in this document.

[2]See Equation (3-20) for the definition of equinoctial elements in terms of the classical elements.

[3]See Equations (3-18) and (3-19) for the definition of orbital quaternions.

Primary reason:

- Attitude control (Earth-pointing missions only)

Secondary reasons:

- Computation of maneuvers for pointing antennas at a Tracking and Data Relay Satellite System (TDRSS) satellite
- Prediction of occultation of a celestial object by the Earth or the Moon
- Reference to the correct segment of the star catalog (Earth-pointing missions)
- Computation of stellar or solar aberration factor due to spacecraft velocity
- Computation of local geomagnetic field for preprocessing star tracker output and during attitude acquisition using Sun sensors and magnetometers
- Computation of dynamic misalignments among onboard sensors
- Annotation of picture and other scientific data gathered by payload sensors

Therefore, the MMS operation will require orbit determination and prediction on the ground and transmission of a representation[1] of the predicted orbit to the satellite. To satisfy MMS mission requirements, the representation must agree with the predicted orbit to within about 10 meters for the Earth-pointing

[1]By a representation is meant a method for approximating the original ephemeris, such as a polynomial or a trigonometric series, with a finite number of coefficients. Use of a representation permits transmission of a finite number of terms to the satellite from which the approximate orbital state at any desired time can be recovered using a relatively simple onboard algorithm. Neither integration of dynamical equations nor orbit determination from measurements is called for in this kind of onboard algorithm.

SECTION 2 - EVALUATION OF EPHEMERIS REPRESENTATION ACCURACY

The accuracy of several ephemeris representations is discussed in this section. Both interpolation and extrapolation accuracies are evaluated by comparison with a high precision ephemeris. The interpolators examined include Adams, Lagrange, and Hermite.[1]

Representations in classical and equinoctial elements are evaluated in Section 2.1. The equinoctial elements are a nonsingular set of orbital elements particularly appropriate for the near-circular orbits under consideration (Reference 5). In addition, this set simplifies conversion to the quantities needed in the attitude control algorithm (see Section 3).

Representations in Cartesian coordinates are evaluated in Section 2.2. Previously, orbital representations of Cartesian coordinates were used successfully to support other missions (References 6 and 7). Further, representation of the Cartesian coordinates is expected to be directly compatible with the Global Positioning System (GPS) data (Reference 3 and Appendix C).

Section 2.3 discusses accuracy of an orbital quaternion representation. This representation describes the instantaneous orbital attitude and is expected to minimize the computations required in the EOS attitude control function (see Section 3).

2.1 REPRESENTATIONS USING EQUINOCTIAL AND CLASSICAL ELEMENTS

The accuracy of interpolation and extrapolation on mean elements is evaluated in Section 2.1.1. Section 2.1.2 deals with interpolation and extrapolation on osculating elements.

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[1] All three of these interpolators are related to polynomial representations. A Lagrange interpolator constrains the polynomial to agree with the original function at selected grid points, a Hermite interpolator constrains both the function and its first derivative at all the grid points, while an Adams interpolator constrains the function at one grid point and its first derivative at all remaining grid points.

Table 2-1. Accuracy of Mean Orbital Element Representation for an SMM Type Orbit[a]

Interpolator Type	Grid Interval (days)	Number of Terms (4 days)	Element Type	Spacecraft Area (km^2)	Maximum RSS Error[b]		
					Converted Elements: Any Span (km)	Integrated Elements 4-Day (km)	Integrated Elements 8-Day (km)
Adams 6th Order	1	36	Equinoctial	10^{-5}	8[c]	10	15
	2	36	Equinoctial	10^{-5}	13[c]	N/A[d]	15
	1	36	Equinoctial - Refined Drag Model	10^{-5}	N/A	7	N/A
	1.5	36	Equinoctial	10^{-5}	N/A	12.5	19.3
Adams 12th Order	1	72	Equinoctial	10^{-5}	N/A	10.3	N/A
	1	72	Equinoctial - Refined Drag Model	10^{-5}	N/A	7.7	N/A
Lagrange	1	30	Equinoctial	10^{-5}	6.6	10.4	N/A
	2	30	Equinoctial	10^{-5}	6.8	N/A	N/A
	2	30	Equinoctial	10^{-6}	6.3	N/A	N/A
	2	30	Keplerian	10^{-6}	6.4	N/A	N/A
Hermite 3-Point	2	36	Equinoctial	10^{-5}	6.7	10.4	N/A

[a] See Appendix A, Table A-1, for the nominal orbital elements.

[b] RSS Error = root sum square of positional component errors.

[c] Not exactly converted elements; one starter iteration was permitted.

[d] N/A = not available.

Evaluation results of mean element representations (Table 2-1) show that orbital errors are of the order of the J_2 perturbation, i.e., around 7 kilometers, when mean elements are used.[1] The extrapolation accuracy is comparable to the interpolation accuracy; in fact, they are identical when converted mean elements are used. Thus, the SMM mission requirement is adequately satisfied by the use of mean elements.

2.1.2 Interpolation and Extrapolation on Osculating Elements

In view of the results of the previous subsection, it is evident that in order to satisfy the 10-meter representation accuracy requirement of the Earth Observation Satellite (EOS) mission, osculating rather than mean elements should be used. Due to the short-period effects present in the osculating elements, the interpolation grid interval in this case will have to be about two orders of magnitude smaller than that used with mean elements. Consequently, the number of terms transmitted every 4 days will be on the order of several thousand as compared with the 30 to 40 terms needed for a mean element representation. On the other hand, the onboard software for interpolation and conversion to a form suitable for use in the onboard computations will be nearly identical for mean and osculating element representations, with the exceptions that some logic to maintain a moving interpolation grid and some special extrapolation scheme will be needed when using osculating elements.

An evaluation of interpolation accuracy using osculating elements was carried out, with emphasis on the EOS mission for which the nominal state vector is given in Table A-1. The results are presented in Table 2.2. Both total and horizontal errors are shown; however, only the horizontal errors contribute to errors in the attitude control of EOS (see Appendix A).

From Table 2.2, it is seen that when using double-precision arithmetic on the IBM System 360, the major part of the positional error is in the radial

[1]Double-precision arithmetic was used on the IBM System 360/75. A final evaluation must use single-precision arithmetic.

2.1.1 Interpolation and Extrapolation on Mean Elements

The orbit may be described by mean orbital elements for lower precision missions, such as star- or Sun-pointing missions with ephemeris representation accuracy requirements of about 10 kilometers. The advantage of using mean elements is that the data requirement is very small. Accordingly, with the Solar Maximum Mission (SMM) in view (see Table A-1 for nominal SMM orbital elements), the accuracy of interpolation and extrapolation on mean elements, classical or equinoctial, was evaluated using a variety of interpolators. The interpolators evaluated were:

- Adams 6th and 12th order interpolators (which use one set of elements and 5 or 11 sets of element rates)
- Lagrange five-point interpolator (which uses five sets of elements)
- Hermite three-point interpolator (which uses three sets of elements and three sets of element rates)

The mean elements and element rates were obtained from the Variation of Parameters (VOP) averaged orbit generator of the Goddard Trajectory Determination System (GTDS). When the integrated mean elements and rates obtained from the averaged orbit generator are used as the grid points, they are referred to as integrated mean elements and rates. Alternatively, a high-precision orbit generator was used to compute osculating elements at the grid points, which were then converted to mean elements using the GTDS numerical osculating-to-mean element conversion procedure. The mean rates corresponding to these elements were computed using the VOP averaged orbit generator. Such elements and rates are referred to as converted mean elements and rates.

In any representation scheme that depends on grid point values, a tradeoff will occur between the accuracy achievable and the number of terms required, which is a function of the grid interval. The grid interval was therefore varied in this and similar studies reported in this section.

Table 2-2. Interpolation Accuracy of Osculating Equinoctial Element Representation

INTERPOLATOR	ORBIT[a]	GRID SPACING (min)	NO. OF TERMS/ 4 DAYS (10^3)	MAXIMUM RSS[b] ERROR (meters)			
				D.P.[c]		S.P.[c]	
				TOTAL	HORIZONTAL	TOTAL	HORIZONTAL
LAGRANGE 5-POINT	SMM	10	3.6	700	N/A[d]	N/A	N/A
	EOS (900km)	6	6	155	N/A	165	N/A
		8	4.5	N/A	N/A	400	N/A
		10	3.6	650	N/A	N/A	N/A
HERMITE 2-POINT	EOS (700km)	2	36	<1	<1	N/A	N/A
		4	18	3	<1	26	23
		6	12	16	3	34	25
		8	9	47	10	57	28
		10	7.2	115	22	128	42
HERMITE 3-POINT	EOS (700km)	6	12	N/A	N/A	N/A	N/A
		8	9	11	1	49	46
		10	7.2	38	2	60	58
		12	6	112	4	114	59
		16	4.5	496	15	N/A	N/A
HERMITE 4-POINT	EOS (700km)	8	7.2	<1	<1	17	15
		10	7.2	1	1	33	18
		12	6	56	<1	59	17
		16	4.5	N/A	N/A	360	33

[a] NOMINAL ORBITAL ELEMENTS ARE GIVEN IN APPENDIX A, TABLE A-1.

[b] RSS ERROR: ROOT SUM SQUARE OF POSITIONAL COMPONENT ERRORS.

[c] D.P.: DOUBLE-PRECISION ARITHMETIC ON THE IBM S-360/75
S.P.: SINGLE-PRECISION ARITHMETIC ON THE IBM S-360/75

[d] N/A: NOT AVAILABLE

direction, while use of single-precision arithmetic degrades the horizontal accuracy to a greater extent than the radial accuracy. Both of these observations may be partially understood by examining the relative fluctuations (from grid point to grid point) in the equinoctial elements.

The radial error is related to errors in the elements a, h, and k (see Equation (3-20) for the definition of equinoctial elements in terms of classical elements). Typical relative fluctuations in these elements for the EOS orbit are about 2×10^{-3}, 5×10^{-1}, and 3×10^{-1}, respectively. For the elements p, q, and L, which contribute to the horizontal error, typical relative fluctuations are 3×10^{-1}, 1×10^{-5}, and 1×10^{-1}, respectively. On the average, the fluctuations are most prominent in h and k. This may be responsible for larger interpolation errors in h and k, leading to larger radial errors when using double-precision arithmetic. When using single-precision arithmetic, the truncation has a larger effect on those elements which are more nearly constant, i.e., q and a (in that order). This may be responsible for the greater deterioration of the horizontal error when using single-precision arithmetic.

The computer on board the MMS will use 18-bit words (see Section 3.2). Double precision on this computer will be slightly better than single precision on the IBM System 360. Thus, at first sight it may appear that the interpolation error figures of interest are the errors in the last two columns of Table 2-2. However, it should be possible to achieve an effective precision intermediate between single and double precision (on the System 360) by proper scaling of the numbers involved. For example, a constant number, corresponding to the average value over all the grid points, could be subtracted, stored in memory, and then added back after interpolation, thereby effectively utilizing core storage space for the varying part of a number only. This should help particularly in the case of the element q and to a lesser extent in a, p, and L. A similar advantage is not available for Cartesian or quaternion representations, since elements of these representations have a zero average value over many grid points.

In view of the above, it may be conjectured that the proper grid spacings for two-point, three-point, and four-point Hermite interpolation on osculating equinoctial elements of the EOS 700-kilometer orbit are about 6, 8, and 10 minutes, respectively, and require 12K, 9K, and 7.2K words of data, respectively. For the SMM mission, the grid spacings could be about twice those for the EOS mission. The Lagrange interpolator was only briefly evaluated, because of its generally larger error.

For the extrapolation of equinoctial elements, a Fourier-power expansion, i.e., a Fourier series with polynomial coefficients, of the form given in Equation (2-2), is a possible representation (see Appendix B). A brief evaluation of this series, with the inclusion of the 19 terms, b_{00}, b_{01}, b_{02}, a_{10}, a_{11}, b_{10}, b_{11}, a_{20}, a_{21}, b_{20}, b_{21}, a_{30}, a_{31}, b_{30}, b_{31}, a_{40}, a_{41}, b_{40}, and b_{41}, showed a root mean square (rms) error of fit of 3.3×10^{-2} kilometers in a, rms errors of fit of 7.6×10^{-6}, 6.1×10^{-6}, 6.5×10^{-6}, and 2.6×10^{-5} in h, k, p, and q, respectively, and an rms error of fit of 7.2×10^{-5} radians in L for the EOS 700-kilometer orbit with a 4 x 4 gravity field and solar, lunar, and drag effects over a 1-day data span using an 8-minute data interval.

An estimate of the closeness of this fit may be derived by an approximate transformation of the errors in the equinoctial elements to errors in position. The along-track position error is roughly the semimajor axis times the angular error in the longitude L, i.e., approximately 0.5 kilometers. The cross-track error is roughly the semimajor axis times the error in i or Ω, both of which can be related to errors in p and q. The errors in i and Ω are found to be about 6×10^{-6} and 2×10^{-5}, respectively, leading to a cross-track error of about 100 meters. The radial error can be related to errors in a, h, and k, resulting in an error of about 70 meters. The net rms positional error is therefore approximately 0.55 kilometers. Additional evaluation of the extrapolation of equinoctial elements was not performed in this study; however, T. Feagin of the University of Tennessee has made such an evaluation (Reference 8), and the results essentially agree with the results obtained here.

Another possible method for the extrapolation of osculating equinoctial elements would be to remove the polynomial behavior by representing the mean element history by a polynomial as in Section 2.1.1. The difference between the osculating and the mean elements could then be represented by a simple Fourier series.

2.2 REPRESENTATIONS USING CARTESIAN COORDINATES

The accuracy of interpolation and extrapolation on inertial Cartesian coordinates is evaluated in Sections 2.2.1 and 2.2.2, respectively.

2.2.1 Interpolation

Given the exact, i.e., predicted, Cartesian coordinates at selected grid points, approximate values at intermediate points can be obtained by polynomial interpolation. The following interpolators were evaluated with respect to an SMM orbit and an EOS orbit (see Appendix A for nominal state vectors):

- Lagrange five-point
- Hermite two-point
- Hermite three-point
- Hermite four-point

Equidistant grid points were used in all interpolators. Results are shown in Table 2-3. The third column indicates the number of terms, i.e., single words[1] on the NSSC, which would have to be transmitted and stored on board per 4-day span. The last column indicates the maximum root sum square (rss) error[2] over any span. (The error was not uniform and a search was generally necessary to obtain the maximum.) It is assumed that interpolation is always restricted to a

[1] Nominally, double words are needed to represent the terms, but a data compression scheme, discussed in Section 2.2.2, may permit single words to represent the terms.

[2] At the time of this evaluation, software to estimate the horizontal component error was not available. However, spot calculations by hand indicate that this component is comparable to the radial error.

Table 2-3. Interpolation Accuracy of Cartesian Coordinate Representation

INTERPOLATOR	ORBIT[a]	GRID SPACING (min)	NO. OF TERMS/ 4 DAYS (10^3)	MAXIMUM RSS[b] ERROR D.P.[c] TOTAL	MAXIMUM RSS[b] ERROR D.P.[c] HORIZONTAL	MAXIMUM RSS[b] ERROR S.P.[c] TOTAL	MAXIMUM RSS[b] ERROR S.P.[c] HORIZONTAL
LAGRANGE 5-POINT	SMM	2	27	100	N/A[d]	N/A	N/A
		4	13.5	110	N/A	N/A	N/A
		6	9	1,000	N/A	N/A	N/A
		10	5.4	12,000	N/A	N/A	N/A
	EOS (900km)	4	13.5	75	N/A	N/A	N/A
		6	9	550	N/A	550	N/A
		10	5.4	7,000	N/A	N/A	N/A
HERMITE 2-POINT	EOS (700km)	2	27	5	N/A	30	N/A
		4	13.5	75	N/A	95	N/A
		6	9	400	N/A	N/A	N/A
		8	6.8	1,000	N/A	N/A	N/A
HERMITE 3-POINT	SMM	4	13.5	1	N/A	N/A	N/A
		10	5.4	200	N/A	N/A	N/A
	EOS (700km)	4	13.5	1	N/A	11	N/A
		6	9	5	N/A	22	N/A
		8	6.8	24	N/A	N/A	N/A
		10	5.4	90	N/A	N/A	N/A
		12	4.5	250	N/A	N/A	N/A
	EOS (900km)	4	13.5	1	N/A	12	N/A
		6	9	5	N/A	13	N/A
		8	6.8	N/A	N/A	27	N/A
		10	5.4	75	N/A	N/A	N/A
							N/A
HERMITE 4-POINT	SMM	6	9	15	N/A	N/A	N/A
		10	5.4	N/A	N/A	28	N/A
		12	4.5	35	N/A	N/A	N/A
		14	3.9	105	N/A	N/A	N/A
		20	2.7	1,300	N/A	N/A	N/A
	EOS (700km)	8	6.8	N/A	N/A	8	N/A
		10	5.4	9	N/A	17	N/A
		12	4.5	33	N/A	N/A	N/A
		16	3.4	340	N/A	N/A	N/A
	EOS (900km)	10	5.4	12	N/A	13	N/A
		12	4.5	15	N/A	19	N/A
		30	1.8	20,000	N/A	N/A	N/A

[a] NOMINAL ORBITAL ELEMENTS ARE GIVEN IN APPENDIX A, TABLE A-1.

[b] RSS ERROR: ROOT SUM SQUARE OF POSITIONAL COMPONENT ERRORS.

[c] D.P.: DOUBLE PRECISION ARITHMETIC ON THE IBM S-360/75
S.P.: SINGLE-PRECISION ARITHMETIC ON THE IBM S-360/75

[d] N/A: NOT AVAILABLE

span bounded by the two central grid points in the case of the Lagrange five-point and the Hermite four-point interpolators and by the extreme grid points in the case of the Hermite two-point and three-point interpolators. Points outside such spans are dealt with in the next subsection, where extrapolation accuracy is examined.

It can be seen from the results in Table 2-3 that, in general, higher order interpolators require smaller smounts of data, i.e., permit larger spacing, than lower order interpolators. A Hermite interpolator permits larger spacing than a Lagrange interpolator of comparable degree. Further, for the same accuracy requirement, the higher, relatively drag-free orbit allows larger grid spacing than the lower, drag-perturbed orbit. However, the actual accuracy requirements are quite distinct for the two types of orbits. The desired grid spacing for an EOS type mission, with a 700-kilometer altitude orbit and a 10-meter representation accuracy requirement, is about 1 to 2 minutes, 1 to 2 minutes, 6 to 8 minutes, and 10 to 12 minutes for the Lagrange five-point, Hermite two-point, Hermite three-point, and Hermite four-point interpolators, respectively. The corresponding figures for an SMM type mission, assumed to have a 500-kilometer altitude orbit and a 10-kilometer representation accuracy requirement, are expected to be about twice the figures for the EOS mission. If the amount of data (or core) is the main consideration, a high-order Hermite interpolator would be optimal. However, if onboard computational time is the main criterion, then a low-order Lagrange or a low-order Hermite interpolator appears preferable (see Section 3). When both core and time aspects are considered together, a low-order Hermite interpolator is found to be preferable to a low-order Lagrange interpolator, since the latter requires more data than the former, while the two require about the same computational time (see Section 3).

2.2.2 Extrapolation

To permit extrapolation in the absence of grid point data, a simple semiempirical function is desirable which approximates the time behavior of the Cartesian coordinates. An additional advantage of such a function is that the grid data transmitted to the satellite need only consist of differences or residuals of the actual grid values and the values computed from this extrapolation function. This should result in considerable savings in the amount of data transmitted, because the magnitude of the residual values would hopefully be several orders smaller than that of the grid Cartesian coordinates themselves.[1] The additional computation needed for evaluating the function and adding the residuals will be negligible, since it will only occur once every grid interval (which, as seen in the previous section, is of the order of a few minutes while the basic attitude control cycle interval is likely to be only 1/8 to 1/4 of a second).

It is shown in Appendix B that a possible semiempirical extrapolation function for a Cartesian coordinate, i.e., an inertial component of either position or velocity, over a span of a few days is of the form

$$\sum_{i=0}^{N} \sum_{j=0}^{M} \left[a_{ij}\, t^j \sin(i\omega t) + b_{ij}\, t^j \cos(i\omega t) \right] \tag{2-1}$$

or explicitly,

$$\begin{aligned} & (b_{00} + b_{01}t + b_{02}t^2 + \cdots\cdots) \\ + & (a_{10} + a_{11}t + a_{12}t^2 + \cdots\cdots) \sin(\omega t) \\ + & (b_{10} + b_{11}t + b_{12}t^2 + \cdots\cdots) \cos(\omega t) \\ + & (a_{20} + a_{21}t + a_{22}t^2 + \cdots\cdots) \sin(2\omega t) \\ + & (b_{20} + b_{21}t + b_{22}t^2 + \cdots\cdots) \cos(2\omega t) + \cdots\cdots \end{aligned} \tag{2-2}$$

which is a Fourier series with polynomial coefficients. Here ω is the fundamental mean orbital frequency. Guidelines for the truncation of this series are discussed

[1]Specifically, if the position (velocity) component residuals are no larger than 13 kilometers (13 meters/second) in magnitude, then each term can be represented by a single NSSC word (18 bits) to a 0.1-meter (0.01-centimeters/second) accuracy.

in Appendix B. A semiempirical procedure would thus consist of determining the coefficients appearing in this series by a least-squares fit (on the ground) to the predicted orbit. The fitting span may be chosen to be the fifth day, when extrapolation is likely to be needed. Over the first 4 days, residuals can be computed with respect to the obtained solution. The data transmitted to the spacecraft would consist of the solution (i.e., the coefficients) and the residuals over the first 4 days.

The preceding procedure would be optimal if (a) the residuals transmitted do not exceed the single word limit and (b) there is no danger of loss or distortion of a residual in transmission or in onboard handling. If condition (a) is not met, then transmitting the actual grid point Cartesian coordinates, rather than residuals, would result in a slight saving in onboard computation cost, or the total span could be segmented and a separate solution obtained for each segment. If condition (b) is not met, then the least-squares fit should be carried out over the entire 5-day span or even possibly over the first 4 days alone, and the expansion given in Equation (2-2) may have to be carried to more terms. This method is essentially an extension of the Block 5D ephemeris storage technique (Reference 6).

Results from an evaluation of the extrapolation accuracy of the technique described above are shown in Table 2-4. Results from a preliminary evaluation using a reduced force model and/or eccentricity, which was made primarily to verify the analysis of Appendix B, are also included here. The last four columns indicate the root mean square (rms) error in the x, y, and z components in the least-squares fit to data at a 4-minute grid interval,[1] and the root sum square (rss) of these rms errors. The maximum error was not examined; however, this must be examined before a final choice of an extrapolation function is made. The maximum error over the last day must meet the extrapolation accuracy requirement, while the maximum error (residual) over the first 4 days must meet the single word requirement.

[1]Except run (a)-10, where an 8-minute interval was used.

Table 2-4. Extrapolation Accuracy of Cartesian Coordinate Representation

ORBIT TYPE[a]	CASE	FORCE MODEL	NO. OF TERMS	TERMS IN SOLUTION[c]	SPAN (hours)	RMS ERROR (km) X	Y	Z	RSS
(a) SMM 450 km ALTITUDE	1	TWO BODY[b]	3	b_{00}, a_{10}, b_{10}	8	$0\ (10^{-8})$	$0\ (10^{-8})$	$0\ (10^{-8})$	$0\ (10^{-8})$
	2	TWO BODY	5	$b_{00}, a_{10}, a_{11}, b_{10}, b_{11}$[d]	8	40.0	40.0	23.0	62.0
	3	TWO BODY	5	$b_{00}, a_{10}, b_{10}, a_{20}, b_{20}$	8	0.6	0.6	0.4	0.9
	4	4 x 4, SUN, MOON	5	$b_{00}, a_{10}, b_{10}, a_{20}, b_{20}, \omega$	8	8.5	8.1	0.9	12.0
	5	4 x 4, SUN, MOON	7	$b_{00}, a_{10}, b_{10}, a_{20}, b_{20}, a_{11}, b_{11}$	8	0.7	1.2	0.8	1.6
	6	4 x 4, SUN, MOON	7	SAME AS IN 5 ABOVE	96	6.7	N/A	N/A	15.0[e]
	7	4 x 4 SUN, MOON, DRAG	13	$b_{00}, b_{01}, b_{02}, a_{10}, a_{11}, a_{12}, b_{10}, b_{11}, b_{12}, a_{20}, b_{20}, a_{30}, b_{30}$	24	2.5	2.9	2.4	4.5
	8	4 x 4, SUN, MOON, DRAG	13	SAME AS IN 7 ABOVE.	120	13.3	24.7	N/A	35.0[e]
	9	4 x 4, SUN, MOON, DRAG	13	$b_{00}, b_{01}, b_{02}, a_{10}, a_{11}, a_{12}, b_{10}, b_{11}, b_{12}, a_{20}, a_{21}, b_{20}, b_{21}$	120	23.8	18.5	21.5	37.0
	10	4 x 4, SUN, MOON, DRAG	13	SAME AS IN 9 ABOVE	96	1.5	9.2	10.8	15.0
(b) EOS 900 km ALTITUDE	1	4 x 4, SUN, MOON	5	$b_{00}, a_{10}, a_{11}, b_{10}, b_{11}$[d]	24	4.0	3.0	5.0	7.0
	2	4 x 4 SUN, MOON	9	$b_{00}, a_{10}, a_{11}, b_{10}, b_{11}, a_{20}, b_{20}, a_{30}, b_{30}; \omega$	24	3.0	N/A	N/A	5.0[e]
	3	4 x 4, SUN, MOON	11	$b_{00}, a_{10}, a_{11}, a_{12}, b_{10}, b_{11}, b_{12}, a_{20}, b_{20}, a_{30}, b_{30}$	24	0.5	0.25	0.5	0.75
	4	4 x 4, SUN, MOON	11	SAME AS IN 3 ABOVE	120	5.0	N/A	N/A	8.0[e]
	5	4 x 4 SUN, MOON	13	SAME AS IN (a) 7	24	0.4	N/A	N/A	0.6[e]
(c) EOS 700 km ALTITUDE	1	4 x 4, SUN, MOON, DRAG	13	SAME AS IN (a) 7	24	0.4	0.15	0.4	0.58
	2	4 x 4, SUN, MOON, DRAG	13	SAME AS IN (a) 7	96	1.5	0.65	1.3	2.2

[a] NOMINAL ORBITAL ELEMENTS FOR THESE ORBITS ARE SHOWN IN APPENDIX A, TABLE A-1.

[b] ECCENTRICITY WAS ZEROED OUT IN THIS RUN.

[c] WHEN THE FREQUENCY WAS NOT IN THE SOLUTION, IT WAS OBTAINED FROM THE AVERAGED VOP ORBIT GENERATOR OR FROM A PREVIOUS RUN WITH FEWER TERMS.

[d] BLOCK 5D FUNCTION

[e] ESTIMATED

It can be seen from the results shown in Table 2-4 that the approach described here is a promising one. The actual terms to be included in the solution must be determined by further studies using the actual orbital elements of the mission under consideration; however, the set of 13 coefficients, b_{00}, b_{01}, b_{02}, a_{10}, a_{11}, a_{12}, b_{10}, b_{11}, b_{12}, a_{20}, a_{30}, b_{20}, and b_{30}, appears to be optimal from the preliminary results presented here.

The rms positional error for the EOS orbit, over a 1-day span, is about 0.6 kilometers when this set of coefficients is used. This corresponds to a relative positional error of about 8×10^{-5}. Since onboard reference to the Fourier-power series is infrequent (i.e., at grid points only) and since the computational cost is relatively small (see Section 3), it should be possible to allow for more terms in the onboard software than are actually needed in this expansion. If, for a particular mission, certain higher order terms are found unnecessary or undesirable due to observability problems, a zero value for these coefficients could be transmitted. Further, if the Fourier-power series itself meets the accuracy requirement of the mission (this could be the case for the SMM mission in view of entry (a)-10 in Table (2-4)), then residuals need not be transmitted and the onboard software can be simply modified to bypass the reading and adding of the residuals. This approach would permit standardization of the onboard software.

2.3 REPRESENTATIONS USING ORBITAL QUATERNIONS

Results of the evaluation of the interpolation and extrapolation accuracy of orbital quaternions are presented in Sections 2.3.1 and 2.3.2, respectively. Because this representation was motivated by the EOS mission attitude control requirements, the relative interpolation accuracy requirement was presumed to be 10^{-6}, which corresponds to about 10 meters in position (see Appendix A). Certain auxiliary quantities, r, $\dot{r}$, and possibly $v_\perp$, the component of velocity perpendicular to the radius vector, must also be transmitted if position and velocity are to be recovered (see Section 3). Interpolation and extrapolation on the quantities r and $v_\perp$ are also evaluated.

2.3.1 Interpolation

Estimates of the interpolation accuracy of quaternions for the EOS mission are presented in Table 2-5. The horizontal error depends on the interpolation error in quaternions, while the radial error depends on interpolation on the auxiliary quantity r. The latter fluctuates between about 7080 kilometers and about 7115 kilometers, while a typical quaternion component fluctuates between 0.75 and -0.75. As a result, the horizontal interpolation error on the NSSC will be comparable to the single-precision error on the IBM S/360, while the radial error on the NSSC will be between the single- and double-precision errors on the IBM S/360. It can be deduced from this discussion that the desired grid spacing for the EOS mission, when using Hermite two-point, three-point, and four-point interpolation on quaternions, is about 6 minutes, 10 minutes, and 15 minutes, respectively, requiring about 10K, 6K, and 4K words of data, respectively. The accuracy for an SMM orbit was not evaluated, but it may be conjectured that grid spacings twice as large as those for the EOS mission can be used.

2.3.2 Extrapolation

A Fourier-power series expansion of the form

$$\begin{aligned}
& \left[A_{10} + A_{11}t + A_{12}t^2 + \cdots\right] \sin\left(\frac{\omega t}{2}\right) \\
+ & \left[B_{10} + B_{11}t + B_{12}t^2 + \cdots\right] \cos\left(\frac{\omega t}{2}\right) \\
+ & \left[A_{20} + A_{21}t + A_{22}t^2 + \cdots\right] \sin(\omega t) \sin\left(\frac{\omega t}{2}\right) \qquad (2\text{-}3) \\
+ & \left[B_{20} + B_{21}t + B_{22}t^2 + \cdots\right] \sin(\omega t) \cos\left(\frac{\omega t}{2}\right) \\
+ & \cdots\cdots
\end{aligned}$$

was used for the extrapolation of quaternion components. Justification for this form is discussed in Appendix B. The use of products such as $\sin(\omega t)\sin[(\omega t)/2]$ instead of harmonics such as $\sin[(3\omega t)/2]$ was for programming convenience only; the two forms are equivalent. For r and $v_{\perp}$, Equation (2-2) was used. In evaluating the rms error of fit, the relative error with respect to the maximum

Table 2-5. Interpolation Accuracy of Quaternion Representation

INTERPOLATOR	ORBIT[a]	GRID SPACING (min)	NO. OF TERMS/ 4 DAYS (10^3)	MAXIMUM RUSS[b] ERROR (meters)			
				D.P.[c]		S.P.[c]	
				TOTAL	HORIZONTAL	TOTAL	HORIZONTAL
HERMITE 2-POINT	EOS (700 km)	4	15	N/A[d]	N/A	23	5
		6	10	55	7	65	10
		8	7.5	173	22	185	24
		10	6	421	54	N/A	N/A
HERMITE 3-POINT	EOS (700km)	8	7.5	2	1.5	23	5.5
		10	6	8	4.5	22	7
		12	5	24	12	29	14
		16	3.8	120	68	N/A	N/A
HERMITE 4-POINT	EOS (700km)	8	7.5	N/A	N/A	10	5
		10	6	3	< 1	10	5
		12	5	N/A	N/A	9	6

[a]NOMINAL ORBITAL ELEMENTS ARE GIVEN IN APPENDIX A, TABLE A-1.

[b]RSS ERROR: ROOT SUM SQUARE OF POSITIONAL COMPONENT ERRORS.

[c]D.P.: DOUBLE-PRECISION ARITHMETIC ON THE IBM S-360/75
S.P.: SINGLE-PRECISION ARITHMETIC ON THE IBM S-360/75

[d]N/A: NOT AVAILABLE

magnitude of the component is relevant. For a typical quaternion component this magnitude is 0.75, for r the magnitude is about 7100 kilometers, and for $v_{\perp}$ the magnitude is about 7.4 kilometers/second. Table 2-6 shows the results of the extrapolation evaluation carried out with respect to an EOS (700-kilometer) orbit and an SMM orbit (see Appendix A, Table A-1, for nominal elements) using a 4 x 4 gravity model and solar, lunar, and drag effects. The data rate used was 8 minutes as opposed to 4 minutes for Cartesian coordinates (Section 2.2.2), in view of the interpolation results (Section 2.3.1). The last six columns of the table show the relative errors in the six components. Examination of Table 2-6 shows that a Fourier-power fit to quaternions requires fewer terms than a fit to Cartesian coordinates to the same accuracy (see Section 2.2.2). This is due primarily to the absence of nonharmonic terms in the expansion of quaternions and is partly due to the larger grid spacing permissible when using quaternions.

The use of the extrapolation function as a data compression scheme requires that residuals over a 4-day span not exceed the NSSC single word size. This requires a relative fit of about 10^{-3} or better. The third entry in Table 2-6 indicates that this requirement is likely to be met by the Fourier-power representation.

Table 2-6. Extrapolation Accuracy of Quaternion Representation

ORBIT TYPE[a]	NO. OF TERMS	TERMS IN SOLUTION[b]	SPAN (hours)	RELATIVE ERROR OF FIT (RMS)					
				q_1	q_2	q_3	q_4	r_c	$v_\perp$
SMM	19	(B_{00}, B_{01}, B_{02}), A_{10}, A_{11}, B_{10}, B_{11}, A_{20}, A_{21}, B_{20}, B_{21}, A_{30}, A_{31}, B_{30}, B_{31}, A_{40}, A_{41}, B_{40}, B_{41}	24	7×10^{-5}	3.3×10^{-4}	4×10^{-5}	2×10^{-4}	8×10^{-6}	9×10^{-6}
LOS (700 km)	19	SAME AS ABOVE	24	5×10^{-5}	2.7×10^{-5}	2.2×10^{-5}	4×10^{-5}	1×10^{-5}	1×10^{-5}
	21	SAME AS ABOVE PLUS A_{12}, B_{12}	96	2.9×10^{-5}	N/A[c]	2.3×10^{-5}	N/A	1.1×10^{-5}	N/A
	9	(A_{00}), A_{10}, A_{11}, B_{10}, B_{11}, A_{20}, B_{20}, A_{30}, B_{30}	18	4×10^{-5}	N/A	3.5×10^{-5}	N/A	N/A	N/A
	11	(A_{00}, A_{01}, A_{02}), A_{10}, A_{11}, B_{10}, B_{11}, A_{20}, B_{20}, A_{30}, B_{30}	24	6×10^{-5}	6×10^{-5}	2.5×10^{-5}	4×10^{-5}	1.1×10^{-5}	1.5×10^{-5}

[a] APPENDIX A, TABLE A-1, LISTS THE NOMINAL ORBITAL ELEMENTS
[b] TERMS IN PARENTHESES USED FOR r AND $v_\perp$ ONLY (SEE EQUATION (3-9))
[c] N/A = NOT AVAILABLE

SECTION 3 - COMPUTATIONAL COST ASPECTS

The computational costs of the various onboard algorithms for interpolation from ephemeris representations and for conversion of the interpolated quantities to a form suitable for onboard use are examined in this section. Algorithms for attitude determination and control are not considered. Section 3.1 gives a summary of the algebraic equations involved. Interpolation and extrapolation, as well as conversion algorithms, are included. In Section 3.2, the computational time on the NSSC of the various algorithms is estimated. The estimates are based only on the arithmetic operations involved; overhead cost is not included. Nevertheless, the comparative speeds of the various approaches considered can be derived from these estimates. A discussion of the core requirements of the various approaches is given in Section 3.3.

3.1 SUMMARY OF ALGORITHMS

The algebraic equations involved in the various interpolation and extrapolation schemes are summarized in Sections 3.1.1 and 3.1.2, respectively. Algorithms for conversion between various forms of orbital description are summarized in Section 3.1.3. In each case, an attempt is made to write the algorithms so as to minimize the computational cost.

3.1.1 Interpolation Algorithms

The onboard computations involved in the Lagrange five-point and Hermite two-, three-, and four-point interpolators are summarized below. In each case, computations that are needed at each grid point and at every interpolation interval are presented. The critical cost arises from the latter type of computation, since grid points are likely to be located several hundred (or even possibly several thousand) interpolation intervals apart.[1] The algorithms were derived from the definitions of Lagrange and Hermite interpolators (see footnote on page 2-1).

[1] For an EOS mission.

3.1.1.1 Lagrange Five-Point Interpolator

(a) Grid Computations

For each element and each rate, the following computations are required:

$$
\begin{aligned}
E &= e_3 \\
D &= \frac{(e_1 - 8e_2 + 8e_4 - e_5)}{12} \\
C &= \frac{(-e_1 + 16e_2 - 30e_3 + 16e_4 - e_5)}{24} \\
B &= \frac{(-e_1 + 2e_2 - 2e_4 + e_5)}{12} \\
A &= \frac{(e_1 - 4e_2 + 6e_3 - 4e_4 + e_5)}{24}
\end{aligned}
\tag{3-1}
$$

Here, e stands for any element or rate and the subscripts refer to the various grid points. The algorithm requires 7n multiplications and 14n additions, where n is the number of elements or rates.

(b) Interpolation Computations

First, the quantity $p = [(t - t_3)/g]$ is computed, where g denotes the grid interval. Then, for each element or rate, the interpolated value is obtained as

$$e = E + p\left\{D + p\left[C + p(B + pA)\right]\right\} \tag{3-2}$$

This computation requires (4n + 1) multiplications and (4n + 1) additions, where n is the number of elements or rates.

3.1.1.2 Hermite Two-Point Interpolator

(a) Grid Computations

For each element-rate pair, the following equations are required:

$$
\begin{aligned}
D &= e_1 \\
C &= g\dot{e}_1 \\
B &= 3(e_2 - e_1) - g(2\dot{e}_1 + \dot{e}_2) \\
A &= 2(e_1 - e_2) + g(\dot{e}_1 + \dot{e}_2)
\end{aligned}
\qquad (3\text{-}3)
$$

The required operations are 3.2n multiplications[1] and 6n additions, where n is the number of element-rate pairs.

(b) Interpolation Computations

First, $p = [(t - t_1)/g]$ is computed. Then, for each element-rate pair, the following expressions are evaluated:

$$
\begin{aligned}
e &= D + p[e + p(B + pA)] \\
\dot{e} &= \frac{[C + p(2B + 3pA)]}{g}
\end{aligned}
\qquad (3\text{-}4)
$$

The required operations are (3j + 4.1k + 1) multiplications and (3j + 2k + 1) additions, where j and k are the number of interpolated elements and rates, respectively. (A savings of 2.1k multiplications is possible if C/g, 2B/g, and 3A/g are stored in memory.)

[1] A multiplication or division by a power of 2 is counted as 0.1 multiplication.

3.1.1.3 Hermite Three-Point Interpolator

(a) Grid Computations

For each element-rate pair, the following computations are performed:

$$
\begin{aligned}
F &= e_2 \\
E &= g\dot{e}_2 \\
D &= e_1 - 2e_2 + e_3 + \frac{g(\dot{e}_1 - \dot{e}_2)}{4} \\
C &= \frac{5}{4}(e_3 - e_1) - \frac{g(\dot{e}_1 + \dot{e}_3 + 8\dot{e}_2)}{4} \\
B &= -\frac{e_1}{2} + e_2 - \frac{e_3}{2} + \frac{g(\dot{e}_3 - \dot{e}_1)}{4} \\
A &= \frac{3(e_1 - e_3) + g(\dot{e}_1 + 4\dot{e}_2 + \dot{e}_3)}{4}
\end{aligned}
\qquad (3\text{-}5)
$$

The required operations are 2.7n multiplications and 17n additions, where n is the number of element-rate pairs.

(b) Interpolation Computations

First, the quantity $p = [(t - t_2)/g]$ is computed. Then, for each element-rate pair, the following computations are performed:

$$
\begin{aligned}
e &= F + p\left\{E + p\left[D + p\left(C + p(B + pA)\right)\right]\right\} \\
\dot{e} &= \frac{E + p\left\{2D + p\left[3C + p(4B + 5pA)\right]\right\}}{g}
\end{aligned}
\qquad (3\text{-}6)
$$

The interpolation operations involve (5j + 7.2k +1) multiplications and (5j + 4k + 1) additions, where j and k are the number of elements and rates interpolated, respectively. (A saving of 3.2k multiplications is possible if E/g, 2D/g, 3C/g, 4B/g, and 5A/g are stored in memory.)

3.1.1.4 Hermite Four-Point Interpolator

(a) Grid Computations

For each element-rate pair, the following computations are required:

$$Y_{1\times 6} = (e_1, e_2, e_3, e_4)A + g(\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4)B \tag{3-7}$$

where A and B are the 4 x 6 matrices

$$A = \begin{bmatrix} \frac{56}{108} & \frac{-124}{108} & \frac{50}{108} & \frac{53}{108} & \frac{-52}{108} & \frac{11}{108} \\ -\frac{11}{4} & \frac{1}{4} & \frac{10}{4} & -\frac{1}{2} & -\frac{3}{4} & \frac{1}{4} \\ 2 & 1 & -\frac{5}{2} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ \frac{25}{108} & \frac{-11}{108} & \frac{-50}{108} & \frac{22}{108} & \frac{25}{108} & \frac{-11}{108} \end{bmatrix} \tag{3-8a}$$

$$B = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} & \frac{1}{36} & \frac{7}{36} & -\frac{5}{36} & \frac{1}{36} \\ -1 & -\frac{7}{4} & 2 & \frac{1}{2} & -1 & \frac{1}{4} \\ -1 & 0 & \frac{7}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ -\frac{1}{18} & \frac{1}{36} & \frac{1}{9} & -\frac{1}{18} & -\frac{1}{18} & \frac{1}{36} \end{bmatrix} \tag{3-8b}$$

When written out explicitly, these equations involve 32n multiplications and 41n additions, where n is the number of element-rate pairs.

(b) Interpolation Computations

First, the quantity $p = [(t - t_2)/g]$ is computed. Then, for each element-rate pair, the following expressions are evaluated:

$$e = e_2 + p\left\{g\dot{e}_2 + p\left[Y_1 + p(Y_2 + p(Y_3 + p(Y_4 + p(Y_5 + pY_6))))\right]\right\}$$

$$\dot{e} = \dot{e}_2 + \frac{p\left\{2Y_1 + p\left[3Y_2 + p(4Y_3 + p(5Y_4 + p(6Y_5 + 7pY_6)))\right]\right\}}{g} \qquad (3\text{-}9)$$

This algorithm requires (7j + 10.2k + 1) multiplications and (7j + 5k + 1) additions, where j and k are the number of interpolated elements and rates, respectively. (A savings of 5.2k multiplications is possible if $2Y_1/g$, $3Y_2/g$, $4Y_3/g$, $5Y_4/g$, $6Y_5/g$, and $7Y_6/g$ are stored in memory.)

3.1.2 EXTRAPOLATION ALGORITHMS

A Fourier-power series has been suggested (see Section 2) as an extrapolation function for Cartesian coordinates, orbital quaternions, or equinoctial elements. This series need only be evaluated at grid points, and therefore its computational cost is not very critical. Nevertheless, for completeness, it is evaluated here.

From a computational point of view, it is advantageous to write such a series as

$$\begin{aligned} e = &\left\{B_{00} + t\left[B_{01} + t(B_{02} + \cdots))\cdots\right]\right\} \\ &+ \left\{A_{10} + t\left[A_{11} + t(A_{12} + \cdots))\cdots\right]\right\} \sin(\omega t) \\ &+ \left\{B_{10} + t\left[B_{11} + t(B_{12} + \cdots))\cdots\right]\right\} \cos(\omega t) \\ &+ \left\{A_{20} + t\left[A_{21} + t(A_{22} + \cdots))\cdots\right]\right\} \sin^2(\omega t) \\ &+ \left\{B_{20} + t\left[B_{21} + t(B_{22} + \cdots))\cdots\right]\right\} \sin(\omega t)\cos(\omega t) \\ &+ \left\{A_{30} + t\left[A_{31} + t(A_{32} + \cdots))\cdots\right]\right\} \sin^3(\omega t) \\ &+ \left\{B_{30} + t\left[B_{31} + t(B_{32} + \cdots))\cdots\right]\right\} \sin^2(\omega t)\cos(\omega t) \\ &+ \cdots\cdots \end{aligned} \qquad (3\text{-}10)$$

where e is any element or rate.

This form is obtained by making use of the fact that trigonometric functions of multiple angles are simple polynomials in the functions of the fundamental angle. For quaternions, the terms B_{0j} will be absent in the series, and the trigonometric functions will be sine and cosine of $[(\omega t)/2]$ multiplied by various powers of $\sin(\omega t)$.

This computation requires 2 sines/cosines, $[(2N + 1)Mn + 2Nn + 2N - 2]$ multiplications, and $[(2N + 1)Mn + 2Nn]$ additions, where N and M are the highest harmonic and the highest power of t, respectively, and n is the number of elements/rates or position/velocity components.

As a concrete example, if $N = 4$, $M = 3$, and $n = 1$, then 2 sines/cosines, 41 multiplications, and 35 additions are involved. In another example, if $n = 6$ and N and M are the same as in the first example (i.e., $N = 4$ and $M = 3$), then 2 sines/cosines, 216 multiplications, and 210 additions are required. The times needed on the NSSC for these two cases are approximately 14 and 59 milliseconds, respectively (see Table 3-1, page 3-20).

3.1.3 CONVERSION ALGORITHMS

The attitude control law is expected to be formulated in terms of target quaternions (References 4 and 9) (or possibly direction cosines) and target angular velocities about the body axes, i.e., body rates (Reference 10, Equation (4-5)). A pointing maneuver, on the other hand, may require the position and velocity vectors, since a relative vector between two objects is involved. Thus, conversion algorithms among various forms of orbital description may be required.

Described in this subsection are the algorithms for the following conversions: (1) conversion from Cartesian coordinates to direction cosines, quaternions, and body rates; (2) conversion from quaternions to Cartesian coordinates and direction cosines; and (3) conversion from equinoctial elements to quaternions, direction cosines, body rates, and Cartesian coordinates. It is assumed that orbital information is uplinked from the ground. Compatibility with the GPS is discussed in Appendix C.

3.1.3.1 Computation of Direction Cosines, Quaternions, and Body Rates From Cartesian Coordinates

3.1.3.1.1 Direction Cosines

The target inertial attitude of an Earth-pointing satellite can be described by a 3 x 3 direction cosine matrix $A = (a_{ij})$. The rows of matrix A are simply the inertial components of the target x_T, y_T, z_T axes. The convention is followed that the target x_T-axis points away from the center of the Earth towards the satellite,[1] the z_T-axis points towards the instantaneous orbit normal, and the y_T-axis completes a right-handed system, so that it is approximately along the satellite velocity vector. The A matrix is given by the following algorithm.

The inverse of the magnitude of the position vector is computed as

$$r^{-1} = (r^2)^{-1/2} \approx \frac{63}{256} r_0^{-1} \left\{ \frac{256}{63} + \Delta\left[-\frac{128}{63} + \Delta\left(\frac{32}{21} + \Delta\left(-\frac{80}{63} + \Delta\left(\frac{10}{9} - \Delta\right)\right)\right)\right]\right\} \quad (3\text{-}11)$$

where $r^2 = (x^2 + y^2 + z^2)$, $\Delta \equiv (r_0^{-2})(r^2) \lesssim 2e$, and $\left[\frac{63}{256} r_0^{-1}\right]$ and r_0^{-2} are stored numbers (r_0, which may be chosen to be the mean semimajor axis, is the approximate value of r). For drag-perturbed orbits, a simple polynomial representation for r_0^{-1} may be necessary so that r_0^{-1} will be updated no more than once every grid point, thus adding negligible cost. The above approximation for the square root results in a 50 to 75 percent saving in time over conventional algorithms. The level of truncation shown here will leave relative errors below 2×10^{-11} for an eccentricity below 0.01; this corresponds to double precision on the NSSC.

---

[1] It is assumed in the text that the satellite is to point towards the Earth's center. The case where it must point towards the subpoint, which is defined to be along a normal to the ellipsoidal Earth figure, is discussed in Appendix D.

The orbital angular momentum vector, $\vec{G}$, is then computed as follows:

$$G_x = y\dot{z} - z\dot{y} \; ; \qquad G_y = z\dot{x} - x\dot{z} \; ; \qquad G_z = x\dot{y} - y\dot{x} \tag{3-12}$$

where x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the inertial Cartesian coordinates of the satellite position and velocity.

Next, the magnitude squared, G^2, and the inverse magnitude, G^{-1}, are determined, where G^{-1} can be found by an algorithm similar to that for r^{-1}. For a drag-free satellite, G^{-1} may be assumed to be constant.

The elements of the direction cosine matrix A are computed as follows:

$$\begin{aligned}
&a_{11} = x r^{-1} ; \qquad a_{12} = y r^{-1} ; \qquad a_{13} = z r^{-1} \\
&a_{31} = G_x G^{-1} ; \qquad a_{32} = G_y G^{-1} ; \qquad a_{33} = G_z G^{-1} \\
&a_{21} = a_{32} a_{13} - a_{33} a_{12} \\
&a_{22} = a_{11} a_{33} - a_{13} a_{31} \\
&a_{23} = a_{12} a_{31} - a_{11} a_{32}
\end{aligned} \tag{3-13}$$

The above computations require 36 multiplications and 24 additions.

3.1.3.1.2 Quaternions

The orbital quaternions, denoted here by $\vec{q} \equiv (q_1, q_2, q_3, q_4)$, can be obtained from the direction cosines by a standard conversion algorithm (Reference 11)

$$\begin{aligned}
&Q = 1 + a_{11} + a_{22} + a_{33} \qquad && q_2 = (a_{21} - a_{12}) q_0 \\
&q_0 = Q^{1/2} / 4 && q_3 = (a_{31} - a_{13}) q_0 \\
&q_1 = 2 q_0 Q && q_4 = (a_{23} - a_{32}) q_0
\end{aligned} \tag{3-14}$$

If Q is less than some preselected tolerance ϵ , a different combination of the diagonal elements is selected, and the subsequent equations are changed accordingly. Subroutine CEULER of the Attitude Data Generation (ADGEN) System may be referred to for a complete algorithm.

The required operations are 1 square root, 4. 2 multiplications, and 6 additions for this algorithm plus 24 additions for the computation of matrix A.

An alternative algorithm which proceeds via equinoctial elements (Reference 5), rather than via direction cosines, is as follows. First, the quantities r^2, r^{-1}, $\vec{G}$, G^2, and G^{-1} are computed as in Section 3.1.3.1.1, which calls for 22 multiplications and 17 additions. Next, the unit vector $\hat{G}$, along the angular momentum and equivalent to the $\hat{w}$ vector in Reference 5, is computed as

$$\hat{G}_x = G_x G^{-1}; \qquad \hat{G}_y = G_y G^{-1}; \qquad \hat{G}_z = G_z G^{-1} \tag{3-15}$$

The vector $\hat{G}$ is also equivalent to the third row of matrix A.

The remaining computation steps of the algorithm are as follows. The quantity G_0 is calculated from

$$G_0 = (1 + \hat{G}_z)^{-1/2} \tag{3-16}$$

via an expansion similar to that used for r^{-1} in Equation (3-11). This procedure is valid since $\hat{G}_z = \cos i$, which is approximately constant and not equal to -1. The following quantities are then computed as

$$\begin{aligned}
G_{00} &= G_0^2 \\
p &= \hat{G}_x G_{00}; \qquad q = -\hat{G}_y G_{00} \\
X_1 &= x - \hat{G}_x(px - qy + z) \\
Y_1 &= y + \hat{G}_y(qy - px - z)
\end{aligned} \tag{3-17}$$

$$c' = X_1 r^{-1} \; ; \qquad s' = Y_1 r^{-1}$$

$$c'' = (1 + c')^{-1/2} \quad (\text{or} \quad s'' = (1 - c')^{-1/2} \quad \text{if} \quad c' \approx -1)$$

$$q' = \frac{(1 + \hat{G}_z) G_0 c''}{2} \quad \left(\text{or} \quad q' = \frac{(1 + \hat{G}_z) G_0 s''}{2}\right) \tag{3-17}$$

(Cont'd)

$$q_3 = s'q' \; ; \qquad q_4 = (1 + c')q' \qquad (\text{or} \quad q_3 = (1 - c')q' \; ; \qquad q_4 = s'q')$$

$$q_1 = q q_4 + p q_3 \; ; \qquad q_2 = p q_4 - q q_3$$

The expressions for the quaternions in Equations (3-17) are equivalent to the following basic expressions in terms of the equinoctial elements p, q, and L:

$$\begin{aligned}
q_1 &= (1 + p^2 + q^2)^{-1/2} \left[q \cos(L/2) + p \sin(L/2) \right] \\
q_2 &= (1 + p^2 + q^2)^{-1/2} \left[p \cos(L/2) - q \sin(L/2) \right] \\
q_3 &= (1 + p^2 + q^2)^{-1/2} \sin(L/2) \\
q_4 &= (1 + p^2 + q^2)^{-1/2} \cos(L/2)
\end{aligned} \tag{3-18}$$

which can be derived from the basic definition of the quaternions in terms of the classical elements (i.e., Ω = longitude of the node, ω = argument of perigee, f = true anomaly, and i = inclination) (Reference 11, Equation (4-4)) given below:

$$\begin{aligned}
q_1 &= \sin(i/2) \cos\left(\frac{\Omega - \omega - f}{2}\right) \\
q_2 &= \sin(i/2) \sin\left(\frac{\Omega - \omega - f}{2}\right) \\
q_3 &= \cos(i/2) \sin\left(\frac{\Omega + \omega + f}{2}\right) \\
q_4 &= \cos(i/2) \cos\left(\frac{\Omega + \omega + f}{2}\right)
\end{aligned} \tag{3-19}$$

In this derivation, the following definitions of the equinoctial elements (Reference 5) are used:

$$
\begin{aligned}
a &= a \\
h &= e \sin(\omega+\Omega) \\
k &= e \cos(\omega+\Omega) \\
p &= \tan(i/2) \sin\Omega \\
q &= \tan(i/2) \cos\Omega \\
L &= \omega+\Omega+f
\end{aligned}
\qquad (3\text{-}20)
$$

where a is the semimajor axis and e is the eccentricity. These elements are nonsingular for all except highly retrograde orbits, i.e., orbits with inclinations near 180 degrees.

This method of computation of quaternions requires 1 square root, 47 multiplications, and 31 additions, compared with the 1 square root, 40 multiplications, and 30 additions required by the first method, which proceeds via the direction cosine matrix A.

3.1.3.1.3 Computation of Body Rates (In Conjunction With Either Direction Cosines or Quaternions)

The body rates are given in terms of the classical elements (Reference 10, Equation (4-5)) as

$$
\begin{aligned}
\omega_x &= \dot{\Omega} \sin i \sin(\omega+f) + \dot{i} \cos(\omega+f) \\
\omega_y &= \dot{\Omega} \sin i \cos(\omega+f) - \dot{i} \sin(\omega+f) \\
\omega_z &= \dot{\Omega} \cos i + \dot{\omega} + \dot{f}
\end{aligned}
\qquad (3\text{-}21)
$$

By manipulation, the above expressions can be reduced to

$$
\begin{aligned}
\omega_x &= r^2 r^{-1} (\hat{G}_x \ddot{x} + \hat{G}_y \ddot{y} + \hat{G}_z \ddot{z}) G^{-1} \\
\omega_y &= 0 \\
\omega_z &= G^2 G^{-1} (r^{-1})^2
\end{aligned}
\qquad (3\text{-}22)
$$

where $\hat{G}_x = a_{31}$, $\hat{G}_y = a_{32}$, and $\hat{G}_z = a_{33}$. The preceding results can also be obtained by noting that ω_z must equal the orbital angular momentum, G, divided by r^2, and that the effect of perturbations is to produce a torque $r(\ddot{\bar{r}} \cdot \hat{G})$ along the orbital y-axis, which should result in a precession of the orbit plane about the x-axis at a rate ω_x = (torque)/(angular momentum) = $r\ddot{\bar{r}} \cdot \hat{G}/G$.

Computation of the body rates requires 8 multiplications and 2 additions.

3.1.3.2 Computations of Direction Cosines, Body Rates, and Cartesian Coordinates From Quaternions

The algorithms for conversion from quaternions to direction cosines, body rates, and Cartesian coordinates are described in this subsection. Conversion to Cartesian coordinates requires certain additional quantities apart from quaternions, as is shown below.

3.1.3.2.1 Conversion to Direction Cosines From Quaternions

The following equations (see Reference 10, Equation (4-15)) are used to convert from quaternions to direction cosines:

$$\begin{aligned}
a_{11} &= q_1^2 - q_2^2 - q_3^2 + q_4^2 \\
a_{12} &= 2(q_1 q_2 + q_3 q_4) \\
a_{13} &= 2(q_1 q_3 - q_2 q_4) \\
a_{21} &= 2(q_1 q_2 - q_3 q_4) \\
a_{22} &= -q_1^2 + q_2^2 - q_3^2 + q_4^2 \\
a_{23} &= 2(q_2 q_3 + q_1 q_4) \\
a_{31} &= 2(q_1 q_3 + q_2 q_4) \\
a_{32} &= 2(q_2 q_3 - q_1 q_4) \\
a_{33} &= -q_1^2 - q_2^2 + q_3^2 + q_4^2
\end{aligned} \tag{3-23}$$

The operations required are 10.6 multiplications and 15 additions.

3.1.3.2.2 Conversion to Body Rates From Quaternions

Assuming that quaternion rates are available, the following equations (see Reference 10, Equation (4-16)) can be used to convert from quaternions to body rates:

$$
\begin{aligned}
\omega_x &= 2(q_4\dot{q}_1 + q_3\dot{q}_2 - q_2\dot{q}_3 - q_1\dot{q}_4) \\
\omega_y &= 2(-q_3\dot{q}_1 + q_4\dot{q}_2 + q_1\dot{q}_3 - q_2\dot{q}_4) \equiv 0^* \\
\omega_z &= 2(q_2\dot{q}_1 - q_1\dot{q}_2 + q_4\dot{q}_3 - q_3\dot{q}_4)
\end{aligned}
\qquad (3\text{-}24)
$$

The operations required are 8.2 multiplications and 6 additions.

3.1.3.2.3 Conversion to Cartesian Coordinates From Quaternions

In order to convert from quaternions to Cartesian coordinates, it is necessary to have available three additional independent quantities which can be chosen to be r, $\dot{r}$, and $v_\perp$ (where $v_\perp$ is defined to be the component of velocity perpendicular to the radius vector and is given in terms of classical elements by $v_\perp = \sqrt{\mu a(1-e^2)}/r$, where μ = the gravitational constant).

First, the first six direction cosines must be found using the first six of the nine equations given by Equation (3-23). Then

$$
\begin{aligned}
&x = r a_{11}; \qquad y = r a_{12}; \qquad z = r a_{13} \\
&\dot{x} = \dot{r} a_{11} + v_\perp a_{21}; \qquad \dot{y} = \dot{r} a_{12} + v_\perp a_{22}; \qquad \dot{z} = \dot{r} a_{13} + v_\perp a_{23}
\end{aligned}
\qquad (3\text{-}25)
$$

give the inertial position and velocity components. This involves 19.4 multiplications and 13 additions (including the computation of the six direction cosines).

The above equations are needed when only quaternions (but not their rates) are available, i.e., if a Lagrange interpolator is used. If a Hermite interpolator is

*This is true only for geocentric rates; for geodetic rates, the equation must be evaluated (see Appendix D).

used, quaternion rates $\dot{\vec{q}}$ will also be available. In this case, the need for $v_{\perp}$ can be dispensed with, as follows:

$$\begin{aligned}
\dot{a}_{11} &= 4(q_1\dot{q}_1 + q_4\dot{q}_4) \\
\dot{a}_{12} &= 2(q_1\dot{q}_2 + q_2\dot{q}_1 + q_3\dot{q}_4 + q_4\dot{q}_3) \\
\dot{a}_{13} &= 2(q_1\dot{q}_3 + q_3\dot{q}_1 - q_2\dot{q}_4 - q_4\dot{q}_2) \\
\dot{x} &= \dot{r}a_{11} + r\dot{a}_{11}\,; \qquad \dot{y} = \dot{r}a_{12} + r\dot{a}_{12}\,; \qquad \dot{z} = \dot{r}a_{13} + r\dot{a}_{13}
\end{aligned} \tag{3-26}$$

Here, four rates, as opposed to one $v_{\perp}$, must be obtained by interpolation, but these rates are also needed in the computation of $\vec{\omega}$. These operations require 21.5 multiplications and 12 additions.

Savings (overlap) occur when A, $\vec{\omega}$, $\vec{r}$, and $\dot{\vec{r}}$ are found together: six direction cosines need not be found twice, saving 10.4 multiplications and 10 additions. With the second method for finding $\dot{\vec{r}}$, the overlap is 12.2 multiplications and 5 additions (8.2 multiplications and 5 additions if $\vec{\omega}$ is not found).

3.1.3.3 Computation of Quaternions, Body Rates, Direction Cosines, and Cartesian Coordinates From Equinoctial Elements

Algorithms for conversion from equinoctial elements to quaternions, body rates, direction cosines, and Cartesian coordinates are described in this subsection.

3.1.3.3.1 Conversion to Quaternions From Equinoctial Elements

The equinoctial elements are defined in terms of classical elements as in Equation (3-20). The expressions for the quaternions given in Equation (3-18) lead to the following algorithms:

$$\begin{aligned}
L' &= L/2 \\
s &= \sin L' \\
c &= \cos L' \\
A' &= (1+p^2+q^2)^{-1/2} \\
q_3 &= sA' \\
q_4 &= cA' \\
q_1 &= qq_4 + pq_3 \\
q_2 &= pq_4 - qq_3
\end{aligned} \tag{3-27}$$

The quantity A' is evaluated by an expansion similar to that given for r^{-1} in Equation (3-11); this expansion is valid since $A' \approx \cos(i/2)$. The preceding computations require 2 sines/cosines, 13.1 multiplications, and 9 additions.

3.1.3.3.2 Conversion to Body Rates (In Conjunction With Quaternions) From Equinoctial Elements

Assuming that $\dot{p}$, $\dot{q}$, and $\dot{L}$ are available, the following computations will give the body rates:

$$
\begin{aligned}
& s' = 2sc \;; \qquad c' = 2c^2 - 1 \;; \qquad A'' = 2(A')^2 \\
& \omega_x = A''(\dot{p}s' + \dot{q}c') \\
& \omega_y = A''(\dot{p}c' - \dot{q}s') \equiv 0^* \\
& \omega_z = \dot{L} + A''(p\dot{q} - q\dot{p})
\end{aligned}
\tag{3-28}
$$

The operations required are 9.3 multiplications and 4 additions when computed in conjunction with the quaternions.

3.1.3.3.3 Conversion to Direction Cosines From Equinoctial Elements

The following equations are used to convert from equinoctial elements to direction cosines:

$$
\begin{aligned}
s' &= \sin L \\
c' &= \cos L \\
Q' &= 1 + p^2 + q^2 \\
Q'' &= Q' B_0 - 1 \\
B &= B_0 \left\{ 1 - Q''\left[1 - Q''(1 - Q''(1 - Q''))\right]\right\}
\end{aligned}
\tag{3-29}
$$

Here, B_0 is the nominal value of $B = (1 + p^2 + q^2)^{-1}$ and is stored in the computer memory. It may be periodically updated to account for any secular change in the inclination.

*This is valid for geocentric control only; the geodetic case is discussed in Appendix D.

The components of the A matrix are computed as follows:

$$
\begin{aligned}
a_{31} &= 2pB; \qquad a_{32} = -2qB; \qquad a_{33} = (2-Q')B \\
a_{23} &= s'a_{31} - c'a_{32} \\
a_{13} &= -c'a_{31} - s'a_{32} \\
a_{11} &= c' + p\,a_{13} \\
a_{12} &= s' + q\,a_{13} \\
a_{21} &= p a_{23} - s' \\
a_{22} &= c' - q\,a_{23}
\end{aligned} \tag{3-30}
$$

Operations required are 2 sines/cosines, 18.2 multiplications, and 14 additions.

3.1.3.3.4 Conversion to Body Rates (In Conjunction With Direction Cosines) From Equinoctial Elements

The following equations are used to convert from equinoctial elements to body rates (in conjunction with the direction cosines):

$$
\begin{aligned}
\omega_x &= 2B(\dot{p}s' + \dot{q}c') \\
\omega_y &= 2B(\dot{p}c' - \dot{q}s') \equiv 0^* \\
\omega_z &= \dot{L} + 2B(p\dot{q} - q\dot{p})
\end{aligned} \tag{3-31}
$$

This requires 6.2 multiplications and 3 additions when computed in conjunction with direction cosines.

3.1.3.3.5 Conversion to Cartesian Coordinates From Equinoctial Elements

The following algorithm is used to convert from equinoctial elements to Cartesian coordinates:

$$
\begin{aligned}
C &= a(1 - h^2 - k^2) \\
D &= C E_0^2 - 1 \\
E &= E_0'\left\{\frac{4}{15} + D\left[-\frac{2}{15} + D\left(\frac{1}{10} - D\right)\right]\right\}
\end{aligned} \tag{3-32}
$$

*This is valid for geocentric control only; the geodetic case is discussed in Appendix D.

where $E \equiv \sqrt{a(1-h^2-k^2)} = \sqrt{a(1-e^2)}$ and $E_0' = \frac{15}{4} E_0$. (E_0 and E_0' are stored nominial values.) The expansion above ignores only terms of the order $J_2^4 \approx 10^{-12}$, while including terms of the order $J_2^3 \approx 10^{-9}$.

The remaining steps of the algorithm to convert from equinoctial elements to Cartesian coordinates follow:

$$
\begin{aligned}
c' &= \cos L \\
s' &= \sin L \\
F &= kc' + hs' \\
r &= C\{1-F[1-F(1-F(1-F(1-F)))]\} \\
p_2 &= p^2 \\
q_2 &= q^2 \\
Q'' &= (1+p^2+q^2)B_0 - 1 \\
B &= B_0\{1-Q''[1-Q''(1-Q''(1-Q''))]\} \\
B' &= rB \\
X_1' &= c'B'; \qquad Y_1' = s'B' \\
S &= \sqrt{\mu}\, EB \\
\dot{X}_1' &= -S(h+s'); \quad \dot{Y}_1' = S(k+c') \\
f_1 &= 1-p_2+q_2; \quad f_2 = 2pq; \qquad f_3 = -2p \\
g_1 &= f_2; \qquad g_2 = 1+p_2-q_2; \quad g_3 = 2q \\
x &= f_1 X_1' + g_1 Y_1' \\
y &= f_2 X_1' + g_2 Y_1' \\
z &= f_3 X_1' + g_3 Y_1' \\
\dot{x} &= f_1 \dot{X}_1' + g_1 \dot{Y}_1' \\
\dot{y} &= f_2 \dot{X}_1' + g_2 \dot{Y}_1' \\
\dot{z} &= f_3 \dot{X}_1' + g_3 \dot{Y}_1'
\end{aligned}
\tag{3-33}
$$

The operations required are 2 sines/cosines, 41.3 multiplications, and 31 additions. Savings (overlap) occur when q, $\vec{\omega}$, $\vec{r}$, $\dot{\vec{r}}$, or A, $\vec{\omega}$, $\vec{r}$, $\dot{\vec{r}}$ are found together. In these cases, s', c', and B need not be recomputed, which saves 2 sines/cosines, 8 multiplications, and 8 additions. (If $\vec{\omega}$ is not found, the savings are 2 sines/cosines, 5 multiplications, and 5 additions.)

3.2 COMPUTATIONAL TIME ESTIMATES

The onboard computational time needed for interpolation and for conversion of the orbital information to a form suitable for use in attitude control and pointing maneuvers is estimated in this section for a variety of ephemeris representations. The estimates reflect only the arithmetic operations as listed in the previous subsections; overhead was not included. The characteristics of the onboard processor which are assumed for these estimates are shown in Table 3-1. (The numbers given in this table are nearly the same as, though not identical to, those in Reference 4.)

Due to the 18-bit word length, which permits only about a 10^{-5} relative accuracy, it is assumed that all operations will be performed in double precision. This yields a relative accuracy of 10^{-10} to 10^{-11}. This is considerably above that required, which is about 10^{-8} for intermediate computations and 10^{-6} for the final results, such as control corrections. Advantage should be taken of this intermediate precision requirement wherever possible (e.g., in expanding roots). This was already done to some extent in writing the algorithms in Section 3.1.

The relevant time estimates are shown in Table 3-2. The numbers listed represent interpolation cycle times in milliseconds. Computations at grid points are not included here because they are infrequent. (These could amount to about 50 to 150 milliseconds per grid interval.)

It is seen from Table 3-2 that if quaternions or direction cosines and body rates are the only quantities required in the attitude control law, it is most efficient to represent the orbit in quaternions, with equinoctial elements offering nearly as efficient a representation and Cartesian coordinates being considerably slower. Conversions to position-velocity for pointing maneuvers are infrequent and therefore should not be a critical factor. (If they were, the Cartesian coordinates would be more suitable.) If, on the other hand, the attitude control law itself requires position and velocity,[1] then Cartesian coordinates are competitive with

[1]This could be the case if geodetic, rather than geocentric, stabilization is desired (Appendix D).

Table 3-1. Characteristics of the Onboard Processor (NSSC)

1. WORD LENGTH -- 18 Bits
2. INSTRUCTION WORD -- 18 Bits
3. DATA FLOW -- Parallel
4. DATA TYPE -- Fixed Point, Fraction, 2's Complement
5. NUMBER OF INSTRUCTIONS -- 55
6. CLOCK RATE/CYCLE TIME -- 800 kHz/1.25 μs*
7. OPERATIONS PER SECOND -- 200K
8. NUMBER OF INDEX REGISTERS -- 1
9. ACCUMULATOR -- Double Length
10. INTERRUPTS -- 16 Multilevel
11. DIRECT MEMORY ACCESS (DMA) -- 16 Devices/channel
12. MAXIMUM I/O RATE -- 66K Words/second
13. COMMAND LOAD AND DUMP -- 300K Words/second
14. MEMORY CAPABILITY -- 4K Word Modules to 64K Words
15. DIRECT ADDRESSING -- 4K Words/Page
16. TECHNOLOGY -- TTL/LSI
17. SIZE/WEIGHT -- (112 cubic inches)/(3 pounds)
18. OPERATION TIMES:

Operation	Single Precision	Double Precision
Add/Subtract	20 μs	65 μs
Multiply†	42 μs	210 μs
Divide†	68 μs	3 ms*
Sine/Cosine	.35 ms	1.5 ms
Square Root/ Inverse Square Root	.66 ms	3-6 ms (Nominal 5 ms)

* μs = microsecond; ms = millisecond

†An exception is the case of multiplication or division by powers of 2; in this case, a single-precision operation will take 4.2 μs and a double-precision operation will take 21 μs.

Table 3-2. Comparative Summary of Computational Costs

REPRESENTATION	QUANTITIES TRANSMITTED PER GRID INTERVAL[a]	ITEMS SAVED IF ω NOT COMPUTED	ITEMS SAVED IF $\vec{r}, \dot{\vec{r}}$ NOT COMPUTED	COMPUTATIONAL TIME FOR QUANTITIES COMPUTED ONBOARD (NEAREST MILLISECOND) [INTERPOLATION TIME (2-POINT/3-POINT/4-POINT HERMITE) + CONVERSION TIME = TOTAL COMPUTATIONAL TIME (WITH A 2- /3- /4-POINT HERMITE INTERPOLATOR)]								
				$\vec{q}$ ONLY	$\vec{q}, \vec{\omega}$ [b]	A ONLY	A, $\vec{\omega}$ [b]	$\vec{r}, \dot{\vec{r}}$	$\vec{q}, \vec{r}, \dot{\vec{r}}$	$\vec{q}, \vec{\omega}, \vec{r}, \dot{\vec{r}}$ [b]	A, $\vec{r}, \dot{\vec{r}}$	A, $\vec{\omega}, \vec{r}, \dot{\vec{r}}$ [b]
QUATERNION	10 ($\vec{q}, \dot{\vec{q}}, r, \dot{r}$)	NONE	2 ($r, \dot{r}$)	4/6/8 + 0 = 4/6/8	8/13/18 + 2 = 10/15/20	4/6/8 + 3 = 7/9/11	8/13/18 + 5 = 13/18/23	9/16/22 + 5[c] = 14/21/27	9/16/22 + 5[c] = 14/21/27	9/16/22 + 7[c] = 16/23/29	9/16/22 + 5[c] = 14/21/27	9/16/22 + 6[c] = 15/22/28
CARTESIAN[d]	9 ($\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}$)	3 ($\ddot{\vec{r}}$)	NONE	6/10/14 + 16 = 22/26/30	9/14/17 + 18 = 27/32/35	6/10/14 + 9 = 15/19/23	9/14/17 + 11 = 20/25/28	6/10/14 + 0 = 6/10/14	6/10/14 + 16 = 22/26/30	9/14/17 + 18 = 27/32/35	6/10/14 + 9 = 15/19/23	9/14/17 + 11 = 20/25/28
EQUINOCTIAL	12 ($a, h, k, p, q, L, \dot{a}, \dot{h}, \dot{k}, \dot{p}, \dot{q}, \dot{L}$)	NONE	6 ($a, h, k, \dot{a}, \dot{h}, \dot{k}$)	3/4/6 + 6 = 9/10/12	6/10/14 + 9 = 15/19/23	3/4/6 + 8 = 11/12/14	6/10/14 + 9 = 15/19/23	5/9/12 + 14 = 19/23/26	5/9/12 + 16 = 21/25/28	8/14/19 + 18 = 26/32/37	5/9/12 + 18 = 23/27/30	8/14/19 + 18 = 26/32/37

[a] THE NUMBERS GIVEN IN THIS COLUMN MAY BE MULTIPLIED BY (6K/GRID INTERVAL IN MINUTES) TO OBTAIN THE NUMBER OF NSSC WORDS FOR A 4-DAY SPAN.

[b] THE ESTIMATES IN THESE CASES CAN BE LOWERED BY 2–4 ms IF CERTAIN ADDITIONAL INTERPOLATION COEFFICIENTS ARE STORED IN MEMORY – SEE SECTION 3.1.

[c] IF THE ADDITIONAL QUANTITIES $v_\perp \equiv \sqrt{\mu a(1-e^2)}/r$, $\dot{v}_\perp$ ARE ALSO TRANSMITTED, THE ESTIMATES IN THESE CASES CAN BE LOWERED BY 3/6/8 ms FOR THE 2- /3- /4-POINT HERMITE INTERPOLATORS.

[d] A 5 POINT LAGRANGE INTERPOLATOR WAS ASSUMED FOR $\ddot{\vec{r}}$.

both quaternions and equinoctial elements (see the last four columns in Table 3-2).

In any event, the final choice among the various forms will be governed additionally by the data transmission requirement. This aspect is discussed in Section 3.3.

3.3 DISCUSSION OF CORE REQUIREMENTS

The current design for the onboard processor does not include any peripheral storage devices. Hence, any data transmitted from the ground will be stored in the main memory and must be included in the onboard core storage cost. The core requirements of various representations are roughly estimated below. Core required for the coding of instructions is not estimated, but this should be negligible compared with the data storage requirements, except when mean elements are used where the data requirement is also negligible.

The number of words required for a 4-day span will approximately equal the number of words per grid point multiplied by 6K and divided by the grid spacing in minutes. The number of words per grid point is 10, 9, and 12 for the quaternion, Cartesian, and equinoctial representations, respectively. It is assumed that each term is a single-precision word representing a residual from an extrapolation function (see Section 2.2.2). Based on the results of the evaluation in Section 2, the core estimates shown in Table 3-3 for the EOS and SMM type missions can be derived. It is assumed here that position, velocity, and body rates, as well as direction cosines (or quaternions), are required on board. If position and velocity are not needed, a core saving of 0 percent, 20 percent, and 50 percent is achieved in the case of Cartesian, quaternion, and equinoctial representations, respectively.

Based on the preliminary results obtained, the quaternion representation is optimal for the EOS mission from the point of view of core requirement and time requirement (as discussed in the previous section). The grid spacings shown in Table 3-3 were estimated in Section 2, taking into account scaling information. Additional study of scaling and of residual magnitude optimization will be required

Table 3-3. Approximate Core Estimates

Orbit Type[a]	Accuracy	Representation	Hermite Interpolation Type	Spacing (min)	Words/ 4 Days
EOS	10-meter (horizontal)	Quaternion	2-point	6	10K
			3-point	10	6K
			4-point	15	4K
		Cartesian	2-point	2	27K
			3-point	6	9K
			4-point	12	4.5K
		Equinoctial	2-point	6	12K
			3-point	8	9K
			4-point	10	7.2K
SMM	10-kilometer (total)	Quaternion	2-point	12	5K[b]
			3-point	20	3K[b]
			4-point	30	2K[b]
		Cartesian	2-point	4	13.5K[b]
			3-point	12	4.5K[b]
			4-point	24	2.3K[b]
		Equinoctial	2-point	1440	120[c]
			3-point	1440	120[c]
			4-point	1440	120[c]

[a]See Appendix A, Table A-1, for the nominal orbital elements.

[b]This is a rough estimate, based on the assumption that grid spacing will be twice the corresponding spacing for the EOS mission. These figures can be reduced to 1K words or less if residuals are not transmitted at all.

[c]Mean elements can be used in this case (see Section 2.1.1).

before a more exact comparison can be made of the core requirements of the three representations.

For the SMM mission, the mean element representation is clearly optimal.[1] If uniformity of software among different MMS missions is desired, then an equinoctial element representation could be used in which short-periodic variations (i.e., osculating-minus-mean elements) could be added on for the EOS mission. If optimization of the EOS mission is an important additional factor, a quaternion representation may be preferable. The extra cost to the SMM mission will then be the time cost of evaluating a Fourier-power series at grid points, which may turn out to be negligible compared with the overall available time budget and the core cost of storing harmonic coefficients and residuals. The coefficients should require 1K or fewer words, while the residuals may require an additional 3K to 5K words. However, if the residuals are not used at all, the net core requirement would be only that of the harmonic coefficients, i.e., 1K words or less. This possibility arises since the Fourier-power series fits the SMM ephemerides to approximately a 15-kilometer accuracy, and further tuning of the series could possibly reduce this error to less than 10 kilometers. In this case, the question of whether to perform a direct series evaluation or to use grid point series evaluation and Hermite interpolation remains open; the answer will depend on the desired frequency of orbital computation.

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[1] However, when the stringent requirement mentioned in footnote 1 on page 1-3 is considered, a Cartesian representation is found to be optimal for the SMM mission.

SECTION 4 - SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

A summary of activities, a summary of results, preliminary conclusions, and recommendations for additional investigation are presented in Sections 4.1, 4.2, 4.3, and 4.4, respectively.

4.1 SUMMARY OF ACTIVITIES

Evaluation of the accuracy and onboard computational cost of several ephemeris representations for the Multimission Modular Spacecraft (MMS) was carried out. Fourier-power and Hermite polynomial representations in Cartesian coordinates, equinoctial elements, and orbital quaternions were examined. Mission constraints of an Earth Observation Satellite (EOS) mission, which are the most stringent among all the MMS missions, were used as the primary evaluation guideline. However, applicability of the ephemeris techniques to other MMS missions, such as the Solar Maximum Mission (SMM), was also examined. Related aspects, such as geodetic control requirements and compatibility with navigation using the Global Positioning System, were also examined briefly.

4.2 SUMMARY OF RESULTS

The following results were derived from this evaluation:

- Although a Fourier-power series is a reasonable representation of either Cartesian, quaternion, or equinoctial element histories over the span of interest (4-5 days), the accuracy of this representation is not adequate to meet the EOS mission requirement and is barely adequate for the SMM mission. Nevertheless, this may be a useful representation for extrapolation purposes, as well as a means of data compression when used in conjunction with polynomial interpolation between closely-spaced grid points. Specifically, a 1-day extrapolation accuracy of about 0.5 kilometers (root mean square (rms)) was achieved for the EOS mission. Further reduction in this error may be possible by the inclusion of more terms in the Fourier-power expansion than those used in this study. The data compression is achieved by transmitting residuals from the Fourier-power series (and coefficients describing the series) instead of actual grid point element values. Indications are that single words on the NASA Standard Spacecraft Computer (NSSC) would suffice to describe the residuals, while double words would be necessary to describe the original elements in order to maintain the desired accuracy of representation.

- A Hermite interpolator that uses exact elements and their rates at closely-spaced grid points is capable of achieving the desired representation accuracy. Specifically, the allowable grid spacing for a 3-point Hermite interpolator is around 8 minutes (slightly smaller for Cartesian coordinates and slightly larger for quaternions) for meeting the EOS accuracy requirement. The amount of data required to be transmitted and stored on board for a 3-day upload is around 6K words (somewhat smaller for quaternions and somewhat larger for Cartesian coordinates and equinoctial elements). The word requirement for the SMM mission is estimated to be about half the above, and is possibly below 1K words if the Fourier-power series can be tuned so as not to require residuals at all.[1] For the SMM mission, in fact, polynomial interpolation on mean equinoctial elements is optimal, as this meets the accuracy requirement and uses only 120 words of data.[1]

- The computational time cost of the Fourier-power series is fairly large when judged in relation to the attitude control cycle. (This is a second reason for excluding the use of such a series in the attitude control cycle.) However, if used at grid points only, the total time contribution of the computation of this series is negligible. The onboard time cost of Hermite interpolation and conversion to quantities

[1]However, see footnote 1 on page 1-3.

needed for the attitude control function is less for quaternion and equinoctial representations (15 to 20 milliseconds)[1] than for a Cartesian representation (about 30 milliseconds). Of course, the relative costs are reversed if Cartesian position and velocity are required (e.g., for scientific data annotation). In this case, the cost is about 10 milliseconds for a Cartesian representation and about 20 to 25 milliseconds when using either of the other two representations. If quaternions, body rates, and position and velocity are desired together (at the same frequency), then a quaternion representation is about 10 milliseconds faster than either of the other two representations.

- A quaternion representation is approximately three to four times faster than either Cartesian coordinates or equinoctial representation if geodetic control is desired.

4.3 CONCLUSIONS

In view of the results cited above, the following may be concluded:

- From the point of view of optimizing the onboard operations of the EOS mission, the preferred ephemeris algorithm uses a quaternion representation in the form of a Fourier-power series and uses residuals at grid points spaced a few minutes apart (with low-order Hermitian interpolation between the grid points).

- Use of quaternion representation for the EOS mission not only optimizes the onboard computational time and core storage needs, but is also the best representation for describing geodetic control. In addition, it can be made compatible with navigation algorithms that use the GPS signals as input.

[1]The numbers quoted here and in the rest of this section assume the use of a three-point Hermite interpolator.

- Uniformity of software among different MMS missions is achievable by the use of the quaternion representation. However, optimality is not necessarily achieved thereby with respect to other MMS missions, such as the SMM, particularly if residuals cannot be dispensed with. To achieve near-uniformity and near-optimality with respect to all missions, an equinoctial element representation is ideal, because mean elements and a large grid spacing (~1 day) could be used for an SMM type mission,[1] and osculating elements and a relatively small grid spacing (~ few minutes) could be used for an EOS type mission. Of course, if uniformity of software is not a requirement, then quaternions could be used for EOS, and mean equinoctial elements[2] could be used for SMM.

4.4 RECOMMENDATIONS

Areas for additional investigation of ephemeris storage techniques for the MMS are as follows:

- Since only a limited investigation of the Fourier-power series representation was possible in the time frame of the present study, an additional evaluation, using more terms in the series, is desirable so as to minimize the extrapolation error and to validate the assumption that single-precision NSSC words will suffice for describing the residuals. Further study of the relative importance of the harmonic and polynomial terms may be worthwhile. It would also be desirable to investigate methods of reducing the number of coefficients to be determined, without sacrificing accuracy of fit. This may be possible by making the series more analytical. For example, using the methods outlined in Appendix B, explicit time behavior arising via the J_2

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[1]However, see footnote 1 on page 1-3.

[2]Cartesian coordinates could also be used (in view of footnote 1 on page 1-3).

secular variations in Ω, ω, etc., may be modeled directly using the analytical orbit propagation theory of Brouwer-Lyddane. Then, only the small departures from this theory need be modeled via polynomial and higher harmonic terms. A further reduction in the number of coefficients may be possible by using independent estimates of the secular rates (such as $\dot{\Omega}$), derived from a Keplerian Variation of Parameters (VOP) orbit generator.

- For the extrapolation of equinoctial elements, a pure Fourier series analysis of osculating minus mean elements would be worth investigating. Again, analytical computation of J_2 short-period oscillations should be investigated, so as to minimize the number of empirically determined coefficients. Another candidate representation for osculating minus mean behavior is a spline fit within one orbital period, with a varying scale to account for the secularly decaying orbital period.

- Considerable effort remains to be expended in optimizing the selected software for actual onboard use. Proper scaling of the quantities involved and other techniques for maximizing the accuracy of the NSSC arithmetic need to be investigated. Candidate ephemeris representations should be reevaluated after effecting such optimization, and in the framework of NSSC precision.

- Investigation should be continued of onboard orbit determination algorithms using the Global Positioning System (GPS) and of their compatibility with the onboard ephemeris storage software adopted.

APPENDIX A - ORBITAL ACCURACY REQUIREMENTS FOR SOME MMS MISSIONS

Most of the initial MMS missions will fall into three basic categories: (1) low-altitude Earth-pointing missions, typified by the Sun-synchronous Earth Observation Satellite (EOS) orbit, with mean altitudes in the range of 700 to 900 kilometers, inclinations of about 99 degrees, and eccentricities of approximately 0.002; (2) star- or Sun-pointing missions, typified by the Solar Maximum Mission (SMM) and the Gamma Ray Explorer (GRE) orbits, with mean altitudes of about 450 to 550 kilometers, inclinations of about 30 degrees, and eccentricities of about 0.02; and (3) geosynchronous Earth-pointing missions, typified by the Synchronous Earth Observatory Satellite (SEOS) orbit.

The prime motivation for imposing an ephemeris accuracy requirement on the EOS mission is to obtain a desired geographic registration accuracy and stability of the Earth pictures taken by the payload sensors. Any uncertainty in the knowledge of the ephemeris contributes in two distinct ways to the picture registration uncertainty:

- Direct contribution to uncertainty in the location of the payload sensor
- Indirect contribution due to uncertainty in the pointing direction of the payload sensor arising from the fact that the ephemeris information is used in the attitude control system. Figure A-1 illustrates these two contributions.

The direct contribution is given by the horizontal component of the spacecraft ephemeris uncertainty (i.e., by the projection of the error ellipsoid in the along-track/cross-track plane), reduced by the ratio of the Earth's radius to the spacecraft semimajor axis, a factor of about 0.9 (this contribution is arc AC in the figure).

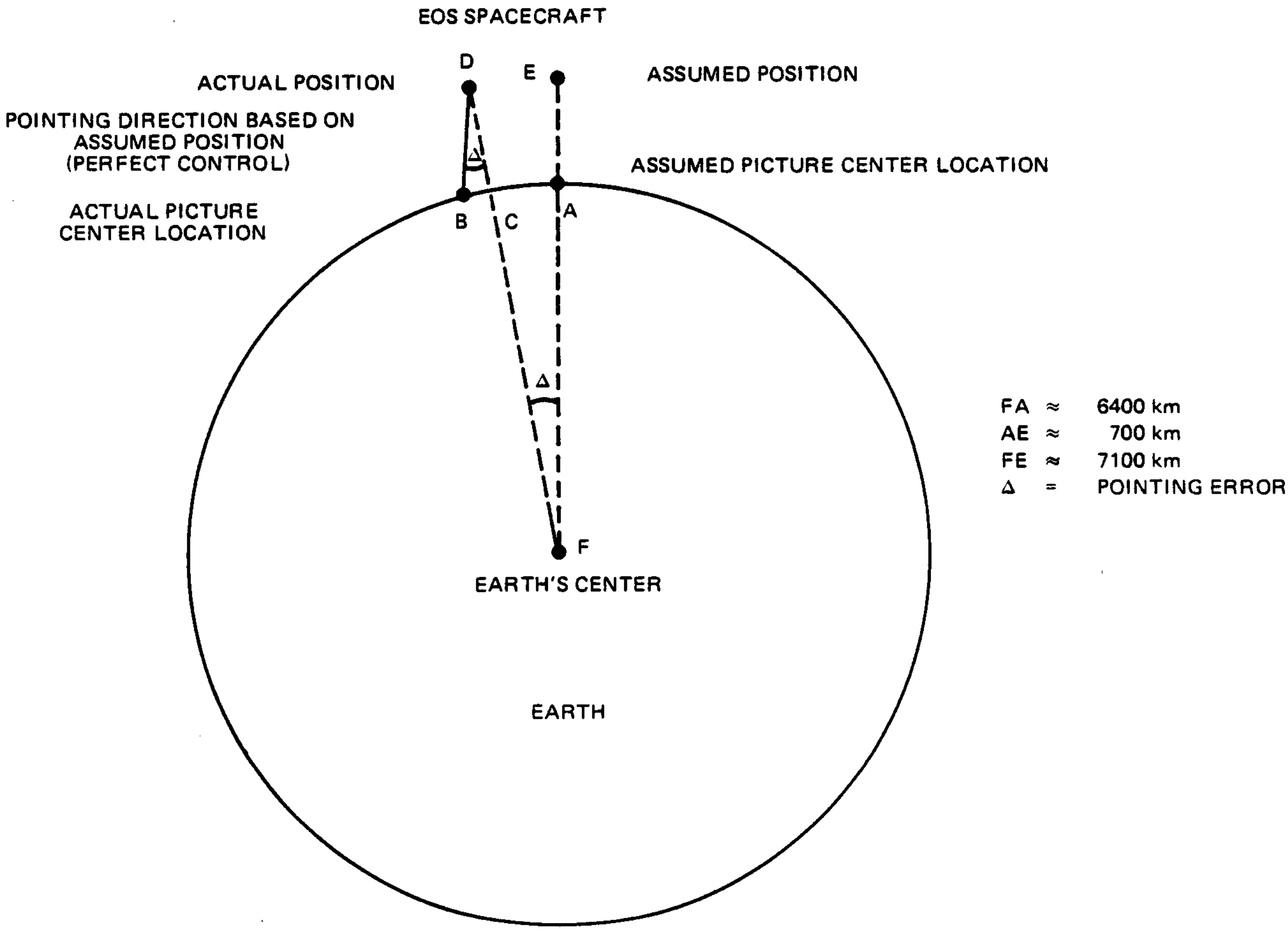

Figure A-1. Geometry of Earth Picture Registration Uncertainty

The indirect contribution, produced by uncertainty in the target direction, is given by the product of the altitude and the uncertainty in the pointing direction. Assuming for simplicity a spherical Earth and an Earth-center-pointing attitude control law, the pointing uncertainty due to the ephemeris uncertainty is the ratio of the horizontal ephemeris uncertainty and the spacecraft semimajor axis (i.e., in Figure A-1, $\Delta = DE/EF$). Thus, this contribution to the picture registration uncertainty equals the horizontal ephemeris uncertainty scaled down by the ratio of the spacecraft altitude to the semimajor axis, which is roughly a factor of 0.1. (This contribution is CB in the figure and equals DB multiplied by Δ, or approximately 0.1 ED.)

If the same ephemerides are used during picture registration as in the attitude control, the net contribution to the picture registration uncertainty is the sum (not a root-sum-square) of the two contributions, i.e., it is given by an ellipse of the same size as the horizontal ephemeris uncertainty. (The net contribution is shown as AB in Figure A-1, and it nearly equals ED.)

In principle, in the absence of operational constraints, unless the ephemeris is determined on board using GPS input (for example), it is not essential that the same ephemerides be used on the ground as on board. There are at least two ways in which the two ephemerides could differ:

- The onboard interpolator need not be as accurate as an interpolator on the ground. For example, the picture registration program could read an ORBIT File using an accurate Adams interpolator, while the onboard software may use a Hermite interpolator with a large grid spacing.

- The ephemerides could conceivably be redetermined during picture registration using a data span (tracking and/or landmark) that covers the picture instead of depending on the predicted orbit that was used on board.

In such cases, the net contribution to the picture registration uncertainty is the root-sum-square of the direct and the indirect contributions. In this case, a relatively large onboard ephemeris error can be tolerated, due to the factor of 0.1 mentioned previously.

As a concrete example, if the horizontal ephemeris uncertainty is 100 meters during picture registration but is 200 meters during onboard attitude control (corresponding to a 6-arc second pointing error), the net registration uncertainty contribution is $\sqrt{90^2 + 20^2} = 92$ meters. A more conservative estimate that accounts for correlations is $\sqrt{100^2 + 10^2} = 101$ meters. Neither of these figures includes the contribution arising from attitude sensor/bias uncertainties. Assuming an 11-arc second figure, an additional contribution of 35 meters must be allowed for, leading to a net uncertainty of $\sqrt{101^2 + 35^2} = 106$ meters.

Throughout the above discussion, the nominal attitude is assumed to be used during picture registration. If the actual attitude is used, the errors are further reduced. In summary, the onboard interpolator can yield considerable errors as long as these errors are not allowed to enter the picture registration programs directly. However, the preferred modes of mission operation foreseen (Reference 12) do not call for redetermination of orbit or attitude on the ground; these parameters are to be telemetered by the spacecraft to the users along with scientific data such as the digital Earth-picture data. In view of this, the onboard ephemeris accuracy requirement for EOS is primarily governed by the picture registration requirement (since the situation described in Figure A-1 holds). This requirement is likely to be in the range of 30 to 100 meters. The prediction accuracy over a 4-day span, obtained using ground processing, may be expected to be around 100 meters for the EOS orbit (Reference 13). Thus, the desired ephemeris representation accuracy is an order of magnitude better, or 10 meters (0.0001 degree or 10^{-6} relative error). In view of the preceding discussion, this requirement may be assumed to hold for the horizontal positional component.

In the case of the SMM and GRE missions, the main purpose of the orbital information is to point an antenna at a TDRSS satellite. Because the TDRSS satellite will be at a geosynchronous altitude, a 0.1-degree pointing accuracy requires about a 70-kilometer position accuracy. Allowing for a factor of 3, the orbital accuracy requirement may be taken to be about 20 kilometers. Because the SMM and GRE orbits are more heavily drag perturbed than the EOS orbits, achievable prediction errors are much larger, i.e., several kilometers over a 4-day span (Reference 14). Thus, the representation accuracy need only be about 10 kilometers (0.1 degree or 10^{-3} relative error) in this case.[1]

The evaluation in this study was carried out with respect to an SMM and an EOS[2] type orbits, keeping in mind 10-kilometer[1] and 10-meter (horizontal) representation accuracy requirements, respectively. The epoch orbital elements and spacecraft parameters used in this evaluation are shown in Table A-1.

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[1] However, see footnote 1 on page 1-3. The requirement mentioned there is as follows. The prediction accuracy over a 3-day span has been estimated in Reference 14 to be about 1 kilometer in the radial and cross-track directions and about 5 kilometers in the along-track component. Assuming that scientific data are to be annotated with the above accuracy, the interpolation accuracy requirement is an order of magnitude better, i.e., 100 meters in the radial and cross-track directions and 500 meters in the along-track component.

[2] In the initial phases of this study, the EOS orbit was assumed to be a 900-kilometer altitude orbit, while in the final phases a 700-kilometer altitude was assumed. These two orbits are referred to as EOS (900 km) and EOS (700 km) orbits, respectively, in this document.

Table A-1. Epoch Orbital Elements and Spacecraft Parameters

Element	SMM Orbit	EOS (900 km) Orbit	EOS (700 km) Orbit
Epoch	710930.	710930.	790228.
Semimajor axis (km)	6832.91[a]	7277.943	7100.
Eccentricity	0.186×10^{-1}	0.334×10^{-2}	0.35×10^{-2}
Inclination (deg)	33.144	99.032	98.9
Longitude of the ascending node (deg)	306.714	147.346	0.
Argument of perigee	66.575	67.251	0.
Mean anomaly (deg)	88.002	188.640	0.
Period (approximate) (min)	94.	103.	99.
Spacecraft area (km^2)	10^{-5} [b]	10^{-5} [b]	10^{-5} [b]
Spacecraft mass (kg)	10^{3} [c]	10^{3} [c]	10^{3} [c]

[a] The actual SMM orbit is likely to be somewhat higher, i.e., around 500 to 550 kilometers; the results of this evaluation are therefore conservative.

[b] This value is a little higher than the planned SMM area of 8.4 $meters^2$ and considerably higher than the area of Landsat-1 of 6.6 $meters^2$. Some runs were made with an area of 10^{-6} $kilometers^2$.

[c] The mass used is slightly smaller than the planned SMM mass of 1043 kilograms and is slightly larger than the Landsat-1 mass of 900 kilograms.

APPENDIX B - DISCUSSION OF SEMIANALYTIC EXTRAPOLATION FUNCTIONS

For a small eccentricity orbit, perturbed by J_2 alone, an expansion of Cartesian coordinates may be carried out in powers of e and of J_2, and the expansion may be truncated at the desired level of accuracy. This process will help determine the required number of harmonics in an ephemeris representation.

Noting that the Cartesian components in the orbital plane are simply $r\cos f$ and $r\sin f$, where f is the true anomaly, the inertial Cartesian coordinates can be written as

$$\vec{X} = a R_3^{-1}(\Omega)\, R_1^{-1}(i)\, R_3^{-1}(\omega) \begin{pmatrix} \frac{r}{a}\cos f \\ \frac{r}{a}\sin f \\ 0 \end{pmatrix} \tag{B-1}$$

where, for example, $R_3(\Omega)$ denotes a rotation angle Ω around the third axis.

The orbit plane coordinates can be expanded in a power series in the eccentricity (see Reference 15, pages 79 and 80), which to order e^2 is

$$\begin{aligned} \frac{r}{a}\cos f &= -\frac{3}{2}e + \left(1 - \frac{3}{8}e^2\right)\cos \ell + \frac{e}{2}\cos 2\ell + \frac{3}{8}e^2\cos 3\ell \\ \frac{r}{a}\sin f &= \left(1 - \frac{5}{8}e^2\right)\sin \ell + \frac{e}{2}\sin 2\ell + \frac{3}{8}e^2\sin 3\ell \end{aligned} \tag{B-2}$$

The mean anomaly, ℓ, is related to time as (Reference 16)

$$\ell = \ell_0 + \omega_0 t + \omega_1 J_2 \sin(2\omega_0 t + q) \tag{B-3}$$

ignoring J_2^2 and higher effects, where ω_0 (the mean orbital frequency), ω_1, ℓ_0, and q are constants.

The factors involving a, ω, and Ω may also be modeled to the first power in J_2, and the net result may be symbolically written as an amplitude as (Reference 16)

$$A = A_0 + A_1 J_2 \sin 2\ell \qquad \text{(B-4)}$$

Combining the preceding equations, a Cartesian component may be symbolically written as

$$X \sim (A_0 + A_1 J_2 \sin 2\ell)(b_0 e + b_1 \sin \ell + b_2 e \sin 2\ell + b_3 e^2 \sin 3\ell) \qquad \text{(B-5a)}$$

i.e.,

$$\begin{aligned} X \sim [A_0 + A_1 J_2 \sin(2\omega_0 t)] [b_0 e &+ b_1 \sin(\omega_0 t) + c_1 J_2 \sin(3\omega_0 t) \\ &+ b_2 e \sin(2\omega_0 t) + c_2 e J_2 \sin(4\omega_0 t) \\ &+ b_3 e^2 \sin(3\omega_0 t) + c_2 e^2 J_2 \sin(5\omega_0 t)] \end{aligned} \qquad \text{(B-5b)}$$

where A_0 and A_1 are of the order of the semimajor axis, and b_0, b_1, b_2, b_3, c_1, and c_2 are of order 1.

Collecting terms and ignoring those terms of orders e^3, J_2^2, and $e^2 J_2^2$ yields

$$\begin{aligned} X \sim (A_0 b_0 e + A_1' b_2' e J_2) &+ (A_0 b_1 + A_1' b_1' J_2) \sin(\omega_0 t) \\ + (A_0 b_2 e + A_1 b_0 e J_2) \sin(2\omega_0 t) &+ (A_0 c_1 J_2 + A_0 b_3 e^2 + A_1' b_1' J_2) \sin(3\omega_0 t) \\ + (A_0 c_2 e J_2 + A_1' b_2' e J_2) \sin(4\omega_0 t) \end{aligned} \qquad \text{(B-6)}$$

where primed quantities are combinations of corresponding unprimed ones. Thus, the order of magnitude of the extrapolation error may be estimated for various harmonics, as shown in Table B-1. For a 0.02 eccentricity and a desired relative error of 10^{-5} (~70 meters), the expansion will have to be carried to at least $3\omega_0 t$, and possibly to $4\omega_0 t$.

Table B-1. Estimated Extrapolation Error for Cartesian Coordinates

Highest Harmonic	Dominant Error Terms	Relative Error Caused			
		e = 0	e = 0.001	e = 0.01	e = 0.1
1	$e,\ J_2,\ eJ_2$	10^{-3}	10^{-3}	10^{-2}	10^{-1}
2	$e^2,\ J_2,\ eJ_2$	10^{-3}	10^{-3}	10^{-3}	10^{-2}
3	$e^3,\ eJ_2,\ J_2^2$	10^{-6}	10^{-6}	10^{-5}	10^{-3}
4	$e^4,\ e^2J_2,\ J_2^2$	10^{-6}	10^{-6}	10^{-6}	10^{-4}

Finally, the model for the amplitude as well as mean anomaly will include polynomial terms in time, since drag and long-period effects[1] are present. These can be effectively included in terms such as t, t^2, $t \sin \omega_0 t$, $t^2 \sin \omega_0 t$, $t \sin 2\omega_0 t$, $t^2 \sin 2\omega_0 t$, etc. The number of such terms required will depend on the span of interest and on orbital and spacecraft parameters.

Thus, the Fourier-power series described in Section 2, Equation (2-1), is seen to be a reasonable semiempirical function for Cartesian coordinates.

A similar analysis of equinoctial elements and quaternions shows that the Fourier-power series expansion is applicable to these forms as well, with certain differences. The semimajor axis, a, has secular drag effects and short-period J_2 effects. The elements h and k have long-period effects due to the presence of

[1]Long-period effects enter via trigonometric functions of Ω and ω, present in $R_3^{-1}(\Omega)$ and $R_3^{-1}(\omega)$. The secular rate in Ω, which is about 1 degree/day for Sun-synchronous orbits, has a dominant effect in Cartesian coordinate amplitude, as a 1-degree rotation of the near-polar EOS orbit plane can cause as much as a 100-kilometer change in the amplitude of the x or y component. The effect of a secular rate in ω is only felt via corrections of the order of J_2.

Ω and ω, secular effects due to the drag effects on eccentricity, and short-period J_2 effects. The elements p and q involve i and Ω and are again subject to long- and short-period effects. As for the true longitude $(L = \Omega + \omega + f)$, in addition to its nearly linear growth at a rate $(\dot{\Omega}_{mean} + \dot{\omega}_{mean} + \omega_0)$, it has short-period effects due to both J_2 and the eccentricity, since f may be written to the third power in e (Reference 15) as

$$f = \ell + \left(2e - \frac{e^3}{4}\right) \sin \ell + \frac{5e^2}{4} \sin 2\ell + \frac{13e^3}{12} \sin 3\ell \tag{B-7}$$

The major difference between equinoctial elements and Cartesian coordinates is that the number of harmonics needed in a, h, k, p, and q is independent of the eccentricity, since only the J_2 short-period effects are to be modeled. Only L involves the effect of eccentricity. Based on the preceding discussion, the extrapolation errors for equinoctial elements can be estimated as shown in Table B-2.

Table B-2. Estimated Extrapolation Error for Equinoctial Elements

Highest Harmonic	Dominant Error Terms	Relative Error Caused			
		e = 0	e = 0.001	e = 0.01	e = 0.1
1	e^2, J_2, eJ_2	10^{-3}	10^{-3}	10^{-3}	10^{-2}
2	e^3, J_2^2, eJ_2	10^{-6}	10^{-6}	10^{-5}	10^{-3}
3	e^4, J_2^2, e^2J_2	10^{-6}	10^{-6}	10^{-6}	10^{-4}

The equinoctial elements are seen to offer an advantage over Cartesian coordinates in the number of harmonics necessary in the Fourier-power expansion.

The quaternions involve sines and cosines of one-half the true anomaly, f. Using the expansion for f given previously, it can be shown that a Fourier-power series for quaternions will involve only odd half-integral multiples of the orbital frequency. (The constant term is absent in quaternions.) Thus, a typical equation for a quaternion is

$$q = \sin(i/2) \cos\left(\frac{\Omega - \omega - f}{2}\right) = \sin(i/2)\left[\cos\left(\frac{\Omega - \omega}{2}\right) \cos(f/2) + \sin\left(\frac{\Omega - \omega}{2}\right) \sin(f/2)\right] \quad \text{(B-8)}$$

The expansion for f (from Equation (B-7)) gives

$$\cos(f/2) = \cos\left[\frac{\ell}{2} + \left(e - \frac{e^3}{8}\right)\sin \ell + \cdots\right]$$

$$\approx \cos(\ell/2) \cos\left[\left(e - \frac{e^3}{8}\right)\sin \ell\right] - \sin(\ell/2) \sin\left[\left(e - \frac{e^3}{8}\right)\sin \ell\right] + \cdots \quad \text{(B-9)}$$

$$\approx \cos(\ell/2)\left[1 - \frac{1}{2}\left(e - \frac{e^3}{8}\right)^2 \sin^2 \ell\right] - \sin(\ell/2)\left[\left(e - \frac{e^3}{8}\right)\sin \ell - \frac{e^3}{6}\sin^3 \ell\right] + \cdots$$

Products such as $\sin(\ell/2) \sin \ell$ and $\sin(\ell/2) \cos 2\ell$ can be written as sums involving sines and cosines of $\ell/2$, $3\ell/2$, $5\ell/2$, etc. When the J_2 short-period effect in quaternions arising from the presence of Ω and ω is combined with the eccentricity effect in f, the following table of estimated extrapolation errors results.

Table B-3. Estimated Extrapolation Error for Quaternions

Highest Harmonic	Dominant Error Terms	Relative Error Caused			
		e = 0	e = 0.001	e = 0.01	e = 0.1
1/2	e, J_2 eJ_2	10^{-3}	10^{-3}	10^{-2}	10^{-1}
3/2	e^2, J_2, eJ_2	10^{-3}	10^{-3}	10^{-3}	10^{-2}
5/2	e^3, J_2^2, eJ_2	10^{-6}	10^{-6}	10^{-5}	10^{-3}
7/2	e^4, J_2^2, e^2J_2	10^{-6}	10^{-6}	10^{-5}	10^{-4}

In brief, equinoctial elements may be the most suitable orbital coordinates for extrapolation via a Fourier-power series.

Finally, in the absence of drag, the power series in time can be replaced by exact trigonometric expressions in Ω and ω, since it is only these long-period effects that the power series in time is supposed to model. The epoch values and mean rates for Ω and ω could be obtained from an averaged Keplerian VOP orbit generator of the Goddard Trajectory Determination System (GTDS). However, polynomial terms in time will be needed to model drag effects.

APPENDIX C - SOFTWARE COMPATIBILITY WITH THE GLOBAL POSITIONING SYSTEM

In examining the ephemeris representations in Section 3, it was implicitly assumed that ephemerides will be determined, predicted, and uplinked to the spacecraft from the ground. An alternative under consideration is to provide the spacecraft with the capability to accept and process input from the Global Positioning System (GPS) (Reference 3) for onboard orbit determination. This appendix discusses methods of interfacing this approach with the onboard attitude control software.

C.1 DETERMINISTIC METHOD

Position can be deterministically computed if four GPS satellites are simultaneously visible. However, this condition cannot be guaranteed prior to 1985, and even after the system is fully operational, geosynchronous satellites at certain longitudes may fail to see four GPS satellites at once (Reference 17). Thus, the onboard orbit determination algorithm will probably have to be an estimator.

If a deterministic algorithm were possible, Cartesian positional components would be known at any desired time (but not into the future, unless an extrapolation scheme were implemented). However, this would still leave open the problem of determining velocity and acceleration components. Assuming that some approximate scheme, e.g., numerical differencing, can be used for velocity and acceleration computations, the easiest way to use the GPS information in the attitude control cycle would be to compute quaternions (or direction cosines) and body rates from the Cartesian coordinates at every interpolation interval.[1] The cost of this computation can be ascertained from the central column of Table 3-2 by subtracting the interpolation cost. This cost is about 11 milliseconds to obtain A and $\vec{\omega}$ and 18 milliseconds to obtain $\vec{q}$ and $\vec{\omega}$, with an additional cost of about 26 milliseconds if geodetic rather than geocentric stabilization is desired (see Appendix D).

[1] In order to implement this technique in practice, the deterministic algorithm will have to be augmented by some extrapolation scheme, since a deterministic fix takes at least 6 seconds.

C.2 REGRESSION METHODS

Since an estimator is more likely to be adopted than the deterministic method, several approaches to building an orbit estimation algorithm, or filter, are examined below.

C.2.1 Regression Algorithm Using Cartesian Coordinates

The most common orbit filter uses Cartesian coordinates. The state may include position, velocity, and even acceleration components, and a simple orbit propagation model such as a Cowell or a Brouwer-Lyddane orbit generator can be included. In this case, all nine Cartesian components can be obtained at any desired time and then converted to A and $\vec{\omega}$ (or $\vec{q}$ and $\vec{\omega}$) as was done with the deterministic algorithm (with costs of 11 or 18 milliseconds as before). However, another possibility is to use the filter to obtain the Cartesian components only at grid points several minutes into the future[1] to compute quaternions or equinoctial elements. The costs in the attitude control cycle in these cases are almost the same as if the quaternions or equinoctial elements had been uplinked from the ground, i.e., about 18 (15) and 19 to 22 (19 to 22) milliseconds, respectively, for obtaining A and $\vec{\omega}(\vec{q}, \vec{\omega})$, with an additional 31 and 37 milliseconds, respectively, for geodetic control (Appendix D). The core advantage of equinoctial elements will be translated into an advantage in the frequency of accessing the GPS orbit filter.

The algorithms for conversion from Cartesian coordinates to quaternions and to $\vec{\omega}$ are described in Section 3.3.1. For obtaining quaternion rates, the following equation can be used:

$$\dot{\vec{q}} = \frac{1}{2} \begin{pmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \\ 0 \end{pmatrix} \qquad \text{(C-1)}$$

[1]This means that the orbit model will have to include J_2 and drag effects, since the orbit filter must propagate to a 100-meter accuracy over a half-hour span.

Conversion from Cartesian coordinates to the equinoctial elements p and q was discussed in the text (Equation (3-17)). To obtain the remaining elements, the following equations can be used in conjunction with the conversion to p and q:

$$L = \sin^{-1}(s') + 2\pi n \tag{C-2a}$$

or

$$L = \tan^{-1}(s'/c') + 2\pi n \tag{C-2b}$$

where s' and c' denote sin L and cos L, respectively, and where n is the correct number of periods from epoch which must be kept track of with appropriate logic. The additional equations needed to obtain the remaining elements are the following:

$$a = \frac{1}{2r^{-1} - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{-1}}$$

$$\vec{e} = -\vec{r}\,r^{-1} - \mu^{-1}(\vec{G} \times \dot{\vec{r}})$$

$$p_2 = p^2 ; \qquad q_2 = q^2$$

$$f_1 = 1 - p_2 + q_2 ; \quad f_2 = 2pq ; \qquad f_3 = -2p$$

$$g_1 = f_2 ; \qquad g_2 = 1 + p_2 - q_2 ; \quad g_3 = 2q$$

$$h = \vec{e} \cdot \vec{g} ; \qquad k = \vec{e} \cdot \vec{f}$$

The equation for a given above is expandable in a binomial series to five terms.

The equinoctial element rates can be found from (Reference 5)

$$\begin{aligned} H &= G^{-1}(\hat{G} \cdot \ddot{\vec{r}}) \\ K &= G_{00} H \\ \vec{Q} &= \ddot{\vec{r}} + \mu (r^{-1})^3 \vec{r} \\ \dot{a} &= 2a^2 \mu^{-1} (\vec{Q} \cdot \dot{\vec{r}}) \\ J &= (qY_1 - pX_1) H \end{aligned} \tag{C-4}$$

$$
\begin{aligned}
\dot{X}_1 &= -\mu G^{-1}(h+s') \\
\dot{Y}_1 &= \mu G^{-1}(k+c') \\
\dot{p} &= K Y_1 ; \qquad \dot{q} = K X_1 ; \qquad \dot{L} = G(r^{-1})^2 + J \\
\hat{f} &= (1+\hat{G}_z)\frac{\vec{f}}{2} \\
\hat{g} &= (1+\hat{G}_z)\frac{\vec{g}}{2} \\
\dot{h} &= \mu^{-1}\left[(2\dot{X}_1 Y_1 - X_1\dot{Y}_1)\hat{f}\cdot\vec{Q} - X_1\dot{X}_1\hat{g}\cdot\vec{Q}\right] + kJ \\
\dot{k} &= \mu^{-1}\left[(2X_1\dot{Y}_1 - \dot{X}_1 Y_1)\hat{g}\cdot\vec{Q} - Y_1\dot{Y}_1\hat{f}\cdot\vec{Q}\right] - hJ
\end{aligned}
\qquad \begin{matrix}\text{(C-4)}\\ \text{(Cont'd)}\end{matrix}
$$

If inverse trigonometric functions and logic to count n, the number of periods, are to be avoided, a modified interface between GPS output and input to the attitude control software may be possible. Instead of the six elements and six rates given in Equation (C-4), seven elements and seven rates can be used, where L and $\dot{L}$ are replaced by s, $\dot{s}$, c, and $\dot{c}$ (where $s = \sin(L/2)$ and $c = \cos(L/2)$). In the grid-point conversion algorithm, the two equations $\dot{s} = c\dot{L}/2$ and $\dot{c} = -s\dot{L}/2$ will be added. The attitude control software will be modified to interpolate on the four elements p, q, s, and c and the four rates $\dot{p}$, $\dot{q}$, $\dot{s}$, and $\dot{c}$, instead of the three elements p, q, and L. Then the equations $s = \sin(L/2)$ and $c = \cos(L/2)$ can be eliminated, and the equation $\dot{L} = 2\dot{s}/c$ (or $\dot{L} = -2\dot{c}/s$, if c is close to 0) will have to be added. The net increase in the attitude cycle time will be about 2/3/4 milliseconds for a 2-/3-/4-point Hermite interpolator. Before adopting this approach, the interpolation and extrapolation accuracy of s, $\dot{s}$, c, and $\dot{c}$ must be evaluated. (These may be similar to those for quaternions.) If the attitude cycle calls for direction cosines rather than quaternions, then $s' = \sin L$ and $c' = \cos L$ could be used in place of $s = \sin(L/2)$ and $c = \cos(L/2)$.

C.2.2 Regression Algorithm Using Equinoctial Elements or Quaternions

Another approach to interfacing GPS input with attitude control software that employs equinoctial elements or quaternions would be to build the orbit filter itself on equinoctial elements or quaternions. This would eliminate the conversion cost from the attitude control software. The impact of this approach on the orbit filter must be analyzed. There are two major aspects to this question: (1) the filter must be supplied with a propagator in equinoctial elements or quaternions, and (2) the observation and its partial derivatives must be expressed in the selected coordinates. Some preliminary observations on these two aspects are offered below.

C.2.2.1 Orbit Propagation

The precision needed in the onboard orbit propagator will depend on the practicable frequency of update using the GPS input. If a high-precision propagator is desired, the numerical integration of a fairly complex force model may be necessary. This could be done in Cartesian coordinates (Cowell integrator), equinoctial elements, or quaternions. The differential equations in the first two types of coordinates are available in the literature (References 5 and 18). One approach for propagation of quaternions is to view the frequencies $\vec{\omega} \equiv \left[\left(\frac{r}{G}\hat{G}\cdot\ddot{\vec{r}}\right),\ 0,\ \frac{G}{r^2}\right]$ as gyro rates and integrate them as is standard practice in attitude systems (Reference 11). Another approach would be to develop equations for the attitude dynamics of orbital motion. In addition, methods of propagating r and $\dot{r}$ should be coupled with integration of quaternions so as to obtain the complete set from which $\vec{r}$ and $\dot{\vec{r}}$ can be recovered, when necessary.

When employing equinoctial elements or quaternions, conversion to Cartesian coordinates will be called for by the force model, since drag effects are most easily expressed in Cartesian coordinates. Finally, a state transition matrix must be computed. Since the cost of numerically integrating variational equations is rather high, an analytic two-body (or two-body and J_2) transition matrix will be desirable. Such a matrix is simplest in equinoctial elements. When integrating in Cartesian coordinates, cross partial derivatives with equinoctial elements are needed to obtain the transition matrix in Cartesian coordinates. When using quaternions, a similar approach via equinoctial elements can be used. Expressions for $\vec{q}$ in terms of p, q, and L (given in Equation (3-18)) and the expression for $\ddot{\vec{q}}$ (given in Equation (C-1)), coupled with the expression for $\vec{\omega}$ in terms of p, q, L, $\dot{p}$, $\dot{q}$, and $\dot{L}$ (given in Equation (3-24)), can be used to obtain the necessary partial derivatives. Alternatively, it might be possible to derive an analytic state transition matrix for $\vec{q}$ from orbit-dynamical equations expressed directly in quaternions.[1] For a zero-eccentricity orbit, with a two-body model, such a matrix is given simply by

$$\Phi(t,t_0) = \begin{pmatrix} \cos\left[\frac{\omega(t-t_0)}{2}\right] & \sin\left[\frac{\omega(t-t_0)}{2}\right] & 0 & 0 \\ -\sin\left[\frac{\omega(t-t_0)}{2}\right] & \cos\left[\frac{\omega(t-t_0)}{2}\right] & 0 & 0 \\ 0 & 0 & \cos\left[\frac{\omega(t-t_0)}{2}\right] & \sin\left[\frac{\omega(t-t_0)}{2}\right] \\ 0 & 0 & -\sin\left[\frac{\omega(t-t_0)}{2}\right] & \cos\left[\frac{\omega(t-t_0)}{2}\right] \end{pmatrix} \tag{C-5}$$

where $\vec{q}(t) = \Phi(t,t_0)\,\vec{q}(t_0)$ and $\omega = G/r^2$ = constant. The above form is obtained either by comparison with the state transition matrix used in attitude systems (Reference 11) or by using expressions for the quaternions in terms of classical or equinoctial elements, as given in Section 3.

For a nonzero eccentricity, the situation is more complex since ω is not constant. In the language of attitude dynamics, the system has a nonconstant moment of inertia r^2. Using the expansion for r/a in powers of the eccentricity (Reference 15), which to e^3 is

$$r/a = 1 + \frac{1}{2}e^2 + \left(-e + \frac{3}{8}e^3\right)\cos \ell + \left(-\frac{1}{2}e^2\right)\cos 3\ell + \left(-\frac{3}{8}e^3\right)\cos 3\ell \tag{C-6}$$

the angular frequency $\omega \equiv G/r^2$ can be written as

$$\omega = \frac{G}{a^2}\left[\left(1 + \frac{1}{2}e^2\right) + \left(2e + \frac{3}{4}e^3\right)\cos \ell + \left(\frac{5}{2}e^2\right)\cos 2\ell + \left(\frac{13}{4}e^3\right)\cos 3\ell\right] \tag{C-7}$$

---

[1]A harmonic oscillator formulation of orbit equations (Reference 19) could possibly be used to advantage for this purpose.

The state equation for $\vec{q}$ partitions, and the equation for q_1 and q_2 is

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\omega}{2} \\ -\frac{\omega}{2} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \equiv A \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{C-8}$$

with a similar equation holding for q_3 and q_4.

Although ω is not a constant, the state transition matrix has the form given previously in Equation (C-5), since $A(t)$ and $\int A(t)\,dt$ commute (Reference 20), except that $\cos\left[\frac{\omega(t-t_0)}{2}\right]$ and $\sin\left[\frac{\omega(t-t_0)}{2}\right]$ are replaced by the cosine and sine of $\int_{t_0}^{t} \frac{\omega(t)}{2}\,dt$, i.e., by the cosine and sine of

$$\frac{G}{2a^2}\left[\left(1+\frac{1}{2}e^2\right)t + \left(2e+\frac{3}{4}e^3\right)\frac{\sin(nt)}{n} + \left(\frac{5}{2}e^2\right)\frac{\sin(2nt)}{2n} + \left(\frac{13}{4}e^3\right)\frac{\sin(3nt)}{3n}\right]_{t_0}^{t} \tag{C-9}$$

where n is the mean notion. Since J_2 is being ignored, the above expression may, for consistency, be truncated at terms of the order e or e^2 (depending on the value of e). Alternatively, if an approximate two-body propagator for quaternions can be developed, the state transition matrix Φ can be obtained simply as (Reference 21)

$$\Phi(t,t_0) = \begin{pmatrix} \phi_{44} & -\phi_{43} & \phi_{42} & -\phi_{41} \\ \phi_{43} & \phi_{44} & -\phi_{41} & -\phi_{42} \\ -\phi_{42} & \phi_{41} & \phi_{44} & -\phi_{43} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{pmatrix} \tag{C-10}$$

with the four independent elements of Φ given by

$$\begin{pmatrix} \phi_{41} \\ \phi_{42} \\ \phi_{43} \\ \phi_{44} \end{pmatrix} = \begin{pmatrix} -q_4(t_0) & -q_3(t_0) & q_2(t_0) & q_1(t_0) \\ q_3(t_0) & -q_4(t_0) & -q_1(t_0) & q_2(t_0) \\ -q_2(t_0) & q_1(t_0) & -q_4(t_0) & q_3(t_0) \\ q_1(t_0) & q_2(t_0) & q_3(t_0) & q_4(t_0) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{pmatrix} \tag{C-11}$$

If the orbit propagation model does not have to be too precise, then several other alternatives exist. An analytical two-body or two-body and J_2 propagation of equinoctial elements or quaternions (and r and $\dot{r}$) could be used. Again, the propagation model for the latter could be derived either via equinoctial elements or by actually developing dynamic equations for orbital quaternions. Another technique would be to use an empirical propagator for equinoctial elements or quaternions of the type used in their extrapolation (see Section 2). The orbit filter will then adjust the coefficients of the Fourier-power series. The advantage of this method is that the effects of harmonics, drag, and orbital eccentricity are included empirically in the propagator, so that its accuracy is somewhat better than that of an analytic two-body propagator. Another advantage is that this technique is equally applicable to geodetic quaternions, for which analytic propagation is cumbersome at best. The disadvantage is that the state size (dimension) is greatly increased, and observability problems may exist. Also, the cost of such an orbit propagator must be compared with the cost of an analytic propagator.

C.2.2.2 Observation Model

Regardless of the form of the orbit generator, the GPS observation model is most simply expressible in Cartesian coordinates. Thus, conversion to Cartesian coordinates, as well as explicit partial derivatives of these forward equations, must be supplied to the observation model. Conversion algorithms from equinoctial elements, from geocentric quaternions and r and $\dot{r}$, and from geodetic quaternions and H and $\dot{H}$ to Cartesian position and velocity are given in Section 3 and Appendix D. Partial derivatives can be obtained from these algorithms by straightforward differentiation. Partial derivatives of Cartesian coordinates with respect to equinoctial elements are given in Reference 5.

APPENDIX D - SOFTWARE ADAPTIBILITY TO GEODETIC CONTROL REQUIREMENT

The onboard computation cost of conversion from various forms of orbital description to a form suitable for use in the attitude control law for an Earth-pointing mission (such as EOS) was examined in Section 3. It was assumed in this analysis that the satellite must point towards the Earth's center, i.e., it must be geocentrically stabilized. Due to the ellipticity of the Earth, however, this is not equivalent to pointing normally to the surface of the Earth. The maximum angular deviation is about 3×10^{-3} radians for the EOS orbit. If geodetic stabilization is desired, for example, in order to minimize atmospheric refraction effects in the picture-taking process by onboard cameras, then a correction must be computed to the direction cosine matrix or quaternions as obtained in the text. This appendix examines the computational cost of applying this correction. This cost will be added to the attitude control cycle cost, since the correction is orbital position dependent.

The correction can be regarded as a rotation by a small angle

$$\Delta \equiv \phi - \delta$$

where ϕ is the geodetic latitude and δ is the inertial declination $= \sin^{-1}(z/r)$ around an axis normal to the plane formed by the inertial z-axis and the satellite's position vector $\vec{r}$. This definition of the geodetic frame differs somewhat from that in the ADGEN System, where the new z-axis is normal to $\vec{r}$ and to the new x-axis. For a perfectly polar orbit, the two definitions coincide. This axis has components $(\frac{y}{\rho}, -\frac{x}{\rho}, 0)$ in the inertial frame, where $\rho \equiv \sqrt{x^2 + y^2} = \sqrt{r^2 - z^2}$. In order to represent the net rotation as two successive rotations, the components of this axis must be found in the orbital frame. This can be done by premultiplying the above vector by the A matrix; the result is

$$\left(0, \ -\frac{r}{\rho} a_{33}, \ \frac{r}{\rho} a_{23}\right) \tag{D-1}$$

The angle of rotation Δ can be expressed as a power series in e_E^2 (i.e., the Earth figure eccentricity squared, which is approximately 6.7×10^{-3}) as

$$\Delta = \Delta_1 e_E^2 + \Delta_2 e_E^4 + O(e_E^6) \tag{D-2}$$

where

$$\begin{aligned} \Delta_1 &= \frac{a_E}{r} \sin\delta \cos\delta \\ \Delta_2 &= \frac{1}{2}\frac{a_E}{r} \sin^3\delta \cos\delta + \frac{a_E^2}{r^2} \sin\delta \cos\delta \left(\cos^2\delta - \sin^2\delta\right) \end{aligned} \tag{D-3}$$

and where a_E = the Earth's equatorial radius $\approx$ 6378.14 kilometers, $\sin\delta = z/r$, and $\cos\delta = \rho/r$. The maximum value of Δ is approximately

$$\Delta_{max} \approx \frac{1}{2}\frac{a_E}{r} e_E^2 \quad (= 3 \times 10^{-3} \text{ radians for an EOS orbit})$$

The terms ignored in the above expansion are of the order of $e_E^6 = 3 \times 10^{-7}$. The expansion was obtained by a series solution to this equation for the geodetic latitude ϕ, obtained by requiring that the normal to the ellipsoid pass through the satellite position, x, y, z,

$$\begin{aligned} &z^2 + (-2\rho z)\tan\phi + \left[z^2(1-e_E^2) + \rho^2 - a_E^2 e_E^4\right]\tan^2\phi \\ &+ \left[-2\rho z(1-e_E^2)\right]\tan^3\phi + \rho^2(1-e_E^2)\tan^4\phi = 0 \end{aligned} \tag{D-4}$$

Knowing the axis of rotation and the angle of rotation, it is straightforward to correct either the quaternions or the direction cosine matrix. The corrected body rates can be obtained in terms of $\dot{\Delta}$ (the derivative of Δ in Equation (D-2)).

The computational cost of the geodetic correction is shown in Table D-1.[1] The second column in each entry indicates the additional quantities that must be

[1]The geodetic frame attitude could also be obtained by using an iterative process (see subroutine XYZPLH of GTDS or subroutine SUBSAT of the ADGEN System). Preliminary tests indicate that the amount of computation involved in this method is considerably more than that in the series expansion method described above.

Table D-1. Computational Cost of Geodetic Correction

Representation	Additional Quantities Transmitted	Quantity Corrected	Number of Arithmetic Operations			Time (milliseconds)		
			Square Roots	Multiplications	Additions	Estimate for Geodetic Correction	Basic Estimate	Total
Quaternion* ($\vec{q}, \dot{\vec{q}}$)	r, $\dot{r}$	$\vec{q}$ Only	1	65	45	21	6	27
		$\vec{q}$ and $\vec{\omega}$	1	100	65	31	15	46
		A Only	1	70	45	22	9	31
		A and $\vec{\omega}$	1	95	60	29	18	47
Cartesian ($\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}$)	None	$\vec{q}$ Only	1	50	25	16	26	42
		$\vec{q}$ and $\vec{\omega}$	1	80	50	25	32	57
		A Only	1	60	35	20	19	39
		A and $\vec{\omega}$	1	90	55	27	25	52
Equinoctial (p, q, L, $\dot{p}$, $\dot{q}$, $\dot{L}$)	a, h, k, $\dot{a}$, $\dot{h}$, $\dot{k}$	$\vec{q}$ Only	1	75	55	24	10	34
		$\vec{q}$ and $\vec{\omega}$	1	120	85	36	19	55
		A Only	1	85	65	27	12	39
		A and $\vec{\omega}$	1	125	115	39	19	58

*The cost is only 1 millisecond if geodetic quaternions are transmitted; then, $\vec{r}$ and $\dot{\vec{r}}$ can be recovered if H and $\dot{H}$ are available.

transmitted specifically for computing the geodetic correction. An additional core storage of about 1K words will be incurred for each quantity over a 4-day span. Columns four through six indicate the arithmetic operations involved in computing the correction. In the last three columns are shown: (1) the corresponding time estimates for the geodetic correction, (2) the basic time estimates (from Table 3-2), and (3) the total contribution to the attitude control cycle. In these estimates, a three-point Hermite interpolator is assumed. It can be seen that the geodetic control requirement adds significantly to the attitude cycle time.

As for the choice of orbital description, quaternions still have a slight edge in terms of time over Cartesian or equinoctial elements, but the proportionate difference is less pronounced.

Another possibility is that the corrected (i.e., geodetic) quaternions and rates could be transmitted to the spacecraft, thereby reducing the onboard cost from 46 to 16 milliseconds. Then, recovery of position and velocity could be accomplished in the following manner when necessary (Reference 18 and subroutine PLHXYZ of GTDS). Two additional quantities, the height H and its rate $\dot{H}$, must also be transmitted in place of r and $\dot{r}$ in the case of geocentric control. The equations given below depend only on the first row of A, and therefore they are valid even when the geodetic frame is defined as in the ADGEN System. The first row of A is determined as

$$\begin{aligned} a_{11} &= q_1^2 - q_2^2 - q_3^2 - q_4^2 \\ a_{12} &= 2\left(q_1 q_2 + q_3 q_4\right) \\ a_{13} &= 2\left(q_1 q_3 - q_2 q_4\right) \end{aligned} \tag{D-5}$$

Next, the intermediate quantity

$$N = a_E \left(1 - e_E^2 \, a_{13}^2\right)^{-1/2} \tag{D-6}$$

is obtained by a series expansion in e_E^2. Then, the expressions

$$N' = N + H; \quad N'' = N' - e_E^2 N$$
$$x = N' a_{11}; \quad y = N' a_{12}; \quad z = N'' a_{13} \tag{D-7}$$

give the position, and the velocity is given by the equations

$$\dot{a}_{11} = 4(q_1\dot{q}_1 + q_4\dot{q}_4)$$
$$\dot{a}_{12} = 2(q_1\dot{q}_2 + q_2\dot{q}_1 + q_3\dot{q}_4 + q_4\dot{q}_3) \tag{D-8}$$
$$\dot{a}_{13} = 2(q_1\dot{q}_3 + q_3\dot{q}_1 - q_2\dot{q}_4 - q_4\dot{q}_2)$$

$$\dot{N} = e_E^2 a_{13} \dot{a}_{13} \frac{N^3}{a_E^2}$$
$$\dot{N}' = \dot{N} + \dot{H} \tag{D-9}$$
$$\dot{N}'' = \dot{N}' - e_E^2 \dot{N}$$

and

$$\dot{x} = a_{11}\dot{N}' + \dot{a}_{11}N'$$
$$\dot{y} = a_{12}\dot{N}' + \dot{a}_{12}N' \tag{D-10}$$
$$\dot{z} = a_{13}\dot{N}'' + \dot{a}_{13}N''$$

The operations required are 40 multiplications and 25 additions; the time required is about 10 milliseconds. To this must be added the cost of interpolation of the 10 quantities $\vec{q}$, $\dot{\vec{q}}$, H, and $\dot{H}$. This cost is about 9/16/22 milliseconds for a 2-/3-/4-point Hermite interpolator.

Of course, before adopting the preceding approach, the interpolation and particularly the extrapolation accuracy of geodetic quaternions must be evaluated. As is seen from the expansion for Δ, for near-polar orbits the geodetic correction may be regarded as a periodic effect of the order of magnitude 3 x (J_2 correction) or about 20 kilometers, with one-half the orbital period, i.e., the same period as for the J_2 correction. (The next higher harmonic has one-fourth the orbital period and an amplitude of 0.4 kilometers.) Thus, any extrapolation model that is designed to incorporate the J_2 effect will probably absorb the geodetic correction as well. Similarly, the desirable interpolation grid interval for geodetic quaternions will probably be nearly the same as that for geocentric quaternions. Finally, a GPS orbit filter based on geocentric quaternions will carry over smoothly to one based on geodetic quaternions (see Appendix C), provided that a Fourier-power series propagator model is used.

REFERENCES

1. Goddard Space Flight Center, X-700-75-140, Low Cost Modular Spacecraft Description, J. Purcell, May 1975

2. R. desJardins, "Mission Operations for the Low Cost Modular Spacecraft," (presented at the Goddard Space Flight Center, Flight Mechanics and Estimation Theory Symposium, Greenbelt, Maryland, April 1974)

3. Naval Surface Weapons Center, SS-GPS-101B, System Specification for the NAVSTAR Global Positioning System (GPS) (Phase I), B. W. Parkinson, Dahlgren Laboratory, Dahlgren, Virginia, April 1974

4. Computer Sciences Corporation, CSC/TM-76/6072, Preliminary Study of Onboard Attitude Control for the Multimission Modular Spacecraft, A. DeMott, February 1976

5. ---, CSC/SD-75/6025, Mathematical Specifications for the Earth Satellite Mission Analysis Program (ESMAP), A. C. Long, W. D. McClain, and P. J. Cefola, June 1975

6. J. D. Vedder, "Representation and Interpolation of Predicted Satellite Ephemerides," Journal of Spacecraft and Rockets, Vol. 12, No. 8, August 1975, pp. 499-501

7. Radio Corporation of America, RCA Document, ISS Flight Ephemeris Table Generation, H. Brandmaier, December 1973

8. T. Feagin, University of Tennessee Space Institute, private communication

9. TRW Systems Group, SS7-16, Subsystem Specification Attitude Control and Determination Subsystem, HEAO-A, November 1974

10. Computer Sciences Corporation, CSC/SD-76/6001, High Energy Astronomy Observatory-A (HEAO-A) Attitude Determination System Specification and Requirements, C. M. Gray, et. al., February 1976

11. ---, 3000-06000-02TR, Attitude Dynamics Data Simulator (ADSIM), Version 3.1, System Description and User Manual, C. M. Gray, M. E. Plett, and L. A. Rodgers, September 1973

REFERENCES (Cont'd)

12. C. E. Velez, "An Approach to Mission Operations in the 1980's - A Case Study: Landsat Follow-on," (presented at the Goddard Space Flight Center, Flight Mechanics and Estimation Theory Symposium, Greenbelt, Maryland, May 1976)

13. A. J. Fuchs, Goddard Space Flight Center, private communication

14. Goddard Space Flight Center, Internal Memorandum, "Orbit Determination and Orbit Prediction Accuracy for the Solar Maximum Mission (SMM)," R. L. DeFazio, August 23, 1974

15. D. Brouwer and G. Clemence, Methods of Celestial Mechanics. New York: Academic Press, 1961

16. Computer Sciences Corporation, 9101-03500-05TM, Analytic Orbit Determination Theory, W. E. Wagner, September 1972

17. Lockheed Missiles and Space Company, LMSC-D491613, Navstar GPS Satellite-to-Satellite Tracking Study, J. O. Beck, et. al., August 1975

18. Goddard Space Flight Center, X-582-76-77, Mathematical Theory of the Goddard Trajectory Determination System, J. O. Cappellari, Jr., C. E. Velez, and A. J. Fuchs (eds.), April 1976

19. Stiefel, E. L. and G. Scheifele, Linear and Regular Celestial Mechanics. New York: Springer-Verlag, 1971

20. K. Ogata, State Space Analysis of Control Systems. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1967, p. 306

21. H. S. Morton, Jr., J. L. Junkins, and J. N. Blanton, "Analytical Solutions for Euler Parameters," (presented at the AAS/AIAA Astrodynamics Conference, Vail, Colorado, July 16-18, 1973; revised January 1974)